Instability and Concentration in the Distribution of Wealth

Ricardo Fernholz[*]
Claremont McKenna College

Robert Fernholz[†]
INTECH

First Draft: January 25, 2012
This Draft: April 27, 2014

Abstract

We consider a setup in which infinitely-lived households face idiosyncratic investment risk and show that in this case the equilibrium distribution of wealth becomes increasingly right-skewed over time until wealth concentrates entirely at the top. The households in our setup are identical in terms of their patience and their abilities, and we assume that there are no redistributive mechanisms—neither explicit in the form of government tax or fiscal policies, nor implicit in the form of limited intergenerational transfers. Our results demonstrate that the presence of such redistributive mechanisms alone ensures the stability of the distribution of wealth over time.

JEL Codes: E21, E24

Keywords: wealth distribution, inequality, incomplete markets

---

A previous version of this paper appeared under the title "Wealth Distribution without Redistribution." We are grateful to Jess Benhabib for extensive feedback and numerous suggestions that substantially improved this paper. We would also like to thank George Akerlof, Richard Baldwin, Pierre-Olivier Gourinchas, and participants at the 2013 Econometric Society Summer Meeting in Los Angeles and the 2013 CEF Conference in Vancouver for their helpful comments. All remaining errors are our own.

[*]Robert Day School of Economics and Finance, Claremont McKenna College, 500 E. Ninth St., Claremont, CA 91711, (909) 607-3795, rfernholz@cmc.edu.

[†]INTECH, One Palmer Square, Princeton, NJ 08542, bob@enhanced.com.

1 Introduction

A notable collection of recent research has described the significant concentration of income and wealth that exists in many countries of the world. According to Atkinson et al. (2011) and Wolff (2012), the richest 1% of households in the United States hold more than 33% of total wealth and earn nearly 25% of total income, with similar numbers observed in other countries as well.[1] Díaz-Giménez et al. (2002) document that the Gini coefficients of the distributions of income and wealth in the United States are equal to 0.55 and 0.80, respectively, and Davies et al. (2011) find similar Gini coefficients for wealth throughout the world. Over the last three decades, Atkinson et al. (2011) find that there has been an increase in the concentration of income in many countries while Wolff (2010) describes a similar though smaller increase in the concentration of wealth in the United States.

Motivated by these stylized facts, we develop a model in which infinitely-lived households face idiosyncratic investment risk, and we examine the dynamic behavior of the distribution of wealth over time. Our goal is to explore these dynamics in the absence of any redistributive mechanisms, so that the outcome of the model is affected only by households' optimal decisions about how much to consume or save and their realized labor and investment incomes. Because we assume that all households are equally patient and have identical abilities, it is luck alone—in the form of high realized investment returns—that creates diverging levels of wealth. In this setting, we show that the equilibrium distribution of wealth is not stationary, and, using recent results in mathematical finance and stochastic portfolio theory, we prove that it becomes increasingly right-skewed over time and tends to a limit in which wealth is concentrated entirely at the top.

Many of the first attempts to account for the right-skewness of the distribution of wealth assumed that households face uninsurable idiosyncratic labor income risk. While this approach has had some empirical success, many of these so-called Bewley models fail to generate high Gini coefficients and heavily right-skewed wealth distributions.[2] Another explanation

[1] Atkinson (2005), Moriguchi and Saez (2008), Piketty (2003), Piketty and Saez (2003), and Saez and Veall (2005) present detailed analyses of income inequality in the United Kingdom, Japan, France, the United States, and Canada, respectively.

[2] Cagetti and De Nardi (2008) and Ljungqvist and Sargent (2004) provide both a discussion and survey of this extensive literature. Some authors have in fact successfully generated right-skewness in this setting by expanding the setup to include extra features such as borrowing constraints, preferences for bequests, and entrepreneurship (Cagetti and De Nardi, 2006; De Nardi, 2004; Quadrini, 2000) or heterogeneous and fluctuating discount rates (Hendricks, 2007; Krussel and Smith, 1998).

for right-skewed distributions of wealth involves uninsurable investment risk and the multiplicative process of wealth accumulation. The assumption that households face idiosyncratic investment risk was first introduced into a macroeconomic model by Angeletos (2007) and Angeletos and Calvet (2006), and has since been incorporated into models of wealth distribution such as Benhabib and Zhu (2009), Benhabib et al. (2011), and Zhu (2010). The primary motivation for this assumption is empirical. Indeed, Benhabib et al. (2011) describe the extensive evidence that exists showing that both private business equity and principal residence ownership are important sources of idiosyncratic investment risk for individuals and households.[3]

We embrace this empirical evidence and show that in an economy populated by dynasties of infinitely-lived households that face uninsurable investment risk, the economy's total wealth concentrates with the wealthiest household over time. This extreme event occurs despite the fact that all households earn identical labor incomes and have identical investment abilities (measured as the expected instantaneous return of risky investments). What drives this asymptotic result, then, is simply the diverging realized rates of investment return earned by different households over time. Crucially, we consider an economy in which there are no redistributive mechanisms, a fact that distinguishes our setup from those of Benhabib et al. (2011, 2014), Champernowne (1953), and Gabaix (2009). It turns out that these distinguishing assumptions are critical, since these other setups generate stationary distributions of wealth while our setup does not. The key contribution of this paper is to precisely characterize the nature of the instability that exists in economies without redistributive mechanisms. Indeed, the central message of our analysis is that in these cases the equilibrium distribution of wealth becomes increasingly right-skewed over time as the wealthiest household accumulates ever larger quantities of wealth relative to the rest of the population.

We interpret redistribution broadly, so that redistributive mechanisms include any process that proportionately affects wealthy households and poor households differently. This includes explicit mechanisms such as government tax or fiscal policies that directly trans-

[3]For example, according to calculations by Bertaut and Starr-McCluer (2002) and Wolff (2012) using data from the 2001 *Survey of Consumer Finances* (SCF), private business equity and the gross value of principal residences make up, respectively, 27% and 28.2% of total U.S. household wealth. These investments are highly volatile, with a standard deviation for the return of housing roughly equal to 15% according to Case and Shiller (1989) and Flavin and Yamashita (2002), and an even larger volatility for the capital gains and earnings on private equity as reported by Moskowitz and Vissing-Jorgensen (2002).

fer income from wealthy to poor households as well as implicit mechanisms such as limited intergenerational transfers of wealth that reduce the total wealth held by wealthy households proportionally more than that of poor households. Although implicit redistributive mechanisms do not involve direct transfers of wealth from wealthy to poor households, their stabilizing effect on the equilibrium distribution of wealth is the same. One of the goals of our paper is to demonstrate this fact.

Much of the heterogeneous-agent macroeconomics literature includes investment income taxes and other explicitly redistributive government policies. These setups provide a good environment in which to compare the effects of various different government policies on the equilibrium distribution of wealth.[4] For our purposes, it is essential to exclude such policies so that we can examine the dynamic behavior of an economy in which there is no redistribution. Our results do, however, have implications for the role of redistributive government tax and fiscal policies in the economy. For example, our findings are consistent with the simultaneous decrease in the progressivity of taxation and increase in the concentration of income and wealth in the United States that has been documented over the past three decades (see Atkinson et al., 2011 and Wolff, 2010). Indeed, our results about the instability of the distribution of wealth in the absence of redistributive mechanisms suggest a potentially important stabilizing role for redistributive income and estate taxes in a manner that is broadly consistent with some of the conclusions of Benhabib et al. (2011) and Diamond and Saez (2011). More research contrasting these possible benefits of taxation with the well-known distortive costs of taxation is warranted.

Another implication of our results relates to the possibility of redistributive intergenerational transfers. If the economy consists of overlapping generations of finitely-lived households, then intergenerational transfers of wealth that proportionately affect wealthy and poor households differently can act as an implicit redistributive mechanism. This requires that wealthy dynasties of households transfer proportionally less wealth to their offspring than poor dynasties of households. Surprisingly, this implies that the absolute size of these transfers is not directly relevant. Indeed, large transfers that are smaller proportions of total wealth for wealthy households are redistributive, while small transfers that are larger proportions of total wealth for wealthy households are not.[5] In the appendix, we alter our basic

[4]In addition to those papers already mentioned above, Benhabib and Bisin (2007), Castañeda et al. (2003), and Stiglitz (1969) provide valuable analyses of the impact of various government policies on the distribution of wealth.

[5]In other words, whether households transfer 99% or 1% of their wealth to their offspring is irrelevant, but

model so that households have finite lifespans and "joy of giving" bequest motives in order to explore how intergenerational transfers may act as an implicit redistributive mechanism. The main result is that such transfers are redistributive only if households receive positive riskless labor incomes and the intensity of their bequest motives is sufficiently low. In these cases, the stability of the distribution of wealth relies on a delicate interaction between positive discounted labor incomes and limited growth of household wealth.

Many setups, including some that share much in common with our setup such as Benhabib et al. (2011) and Zhu (2010), exploit this subtle interaction between positive human wealth and limited growth of total wealth in order to make intergenerational transfers an implicit redistributive mechanism. This is usually accomplished by assuming riskless labor incomes and a low intensity of bequest motives, which together ensure that positive shocks to wealth across generations proportionally affect poor households more than wealthy households. In the appendix, we describe the details behind this process and discuss the implications. In the rest of this paper, however, we abstract away from the implicit redistribution that can exist with overlapping generations since our goal is to investigate the dynamics of the distribution of wealth in the absence of any kind of redistribution. To some extent, then, our main setup extends those setups that feature overlapping generations to consider the implications of intergenerational transfers of wealth that are not implicitly redistributive. When combined with the results of others such as Benhabib et al. (2011) and Zhu (2010), our results imply that stability of the equilibrium distribution of wealth through intergenerational transfers alone relies on those transfers being proportionally smaller for wealthy households. In light of this conclusion, we believe that a more detailed empirical understanding of the characteristics of intergenerational transfers of wealth could yield substantial insight regarding the dynamics of the distribution of wealth.

In addition to analytically characterizing the behavior of the equilibrium distribution of wealth over time, we also consider a number of model parameterizations and present the corresponding simulations. This exercise allows us to observe the behavior of the wealth distribution over time given a range of different assumptions about the underlying characteristics of the economy. Because all households are identical in every way except for their

---

the relative proportion of wealth that wealthy versus poor households transfer to their offspring is relevant. If wealthy households transfer 98% of their wealth to their offspring, then poor households must transfer 99% of their wealth to their offspring for intergenerational transfers to be an implicit redistributive mechanism. Similarly, if wealthy households transfer 1% of their wealth to their offspring, then poor households must transfer 2% of their wealth to their offspring.

realized rates of return on wealth, we find that the only factor that affects the rate at which wealth concentrates at the top is the degree to which households are exposed to uninsurable idiosyncratic investment risk. As households increase their exposure to idiosyncratic risk, the rate at which the distribution of wealth becomes more right-skewed over time also increases. Of course, households' exposure to investment risk depends on the deep parameters of the model, such as households' risk-aversion, the expected excess returns of risky assets relative to a risk-free asset, and the standard deviation of the returns of risky assets. This central role of idiosyncratic risk in generating a rapid increase in wealth inequality highlights the importance of market incompleteness and uninsurable risk in an economy. Much like Benhabib et al. (2011) and Zhu (2010), our conclusions give a renewed significance to the empirical findings of Case and Shiller (1989), Flavin and Yamashita (2002), and Moskowitz and Vissing-Jorgensen (2002), and suggest that more research about the causes and consequences of incomplete markets is warranted. Indeed, our results indicate that any actions or policies that address uninsurable idiosyncratic investment risk have the potential to reduce income and wealth inequality, and slow the pace at which the economy's wealth accumulates at the top.

The main conclusion of our analysis is clear. In the absence of any redistribution, the distribution of wealth is unstable over time and grows increasingly right-skewed until virtually all wealth is concentrated with a single household. This occurs despite the fact that the households in the economy have identical opportunities and identical preferences and abilities. It is important to emphasize that our setup in this paper, in which there is absolutely no redistribution, is intended to describe an important benchmark case rather than to capture the true state of the world except, perhaps, in extreme situations such as pre-revolutionary France (see Carlyle, 1837). In reality, a number of potentially redistributive mechanisms, such as government tax and fiscal policies and limited intergenerational transfers, constantly affect the economy and influence the extent of concentration of wealth at the top. Indeed, our conclusions highlight the importance of these redistributive mechanisms, since it is their presence alone that ensures the stability of the economy and prevents an outcome in which the distribution of wealth is non-stationary and grows increasingly right-skewed over time.

The rest of the paper is organized as follows. Section 2 describes the basic model in which households face idiosyncratic investment risk and presents the main results about the evolution of income and wealth over time. Section 3 provides several simulations that describe the dynamics of wealth over time under different parameterizations of the model. Section 4 concludes. All proofs and their generalizations are provided in Appendix A, and in Appendix B the basic model is altered and the redistributive implications of finitely-lived households and a "joy of giving" bequest motive are discussed.

2 Model

Consider an economy that is populated by $N \in \mathbb{N}$ households that live forever. At each point in time $t \in [0, \infty)$, each household $i = 1, \ldots, N$ solves a savings-consumption problem that determines how much it will consume and how it will invest her savings.

2.1 Consumption and Investment

Households can invest in a risk-free asset that pays a return of r or in an individual-specific asset that is subject to idiosyncratic risk. Uncertainty in this economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We define an N-dimensional Brownian motion $\mathbf{B}(t) = (B_1(t), \ldots, B_N(t))$, $t \in [0, \infty)$, on this probability space and assume that all stochastic processes are adapted to $\{\mathcal{F}_t; t \in [0, \infty)\}$, the augmented filtration generated by $\mathbf{B}$.[6] For all $i = 1, \ldots, N$, we assume that the price of the individual-specific risky asset P_i follows a geometric Brownian motion so that

$$dP_i(t) = \alpha P_i(t)\, dt + \sigma P_i(t)\, dB_i(t), \tag{1}$$

where B_i is a standard Brownian motion, $\alpha > r$ is the expected instantaneous return of this risky asset, and $\sigma > 0$ is the standard deviation of this instantaneous return.[7] All of the Brownian motions B_i are assumed to be independent of each other. Following the heterogeneous-agent macroeconomics literature, we assume that markets are incomplete and hence the risk involved in these N individual-specific assets cannot be shared or pooled across households. As a consequence, each household in the economy faces uninsurable idiosyncratic investment risk.

[6] In order to simplify the notation, we shall omit many of the standard regularity conditions and technical details involved with continuous-time stochastic processes.

[7] This section's main results are easily reproduced in a setting in which the parameters $\alpha(t)$ and $\sigma(t)$ are time-varying and households have log preferences for consumption over time.

Throughout their lives, households supply one unit of labor inelastically and receive labor income equal to λ. At each point in time t, each household $i = 1, \ldots, N$ chooses to invest a fraction $\phi_i(t)$ of its wealth $w_i(t)$ in the risky asset and chooses to consume a quantity $c_i(t)$. We assume that households obtain utility from consumption and that they have utility functions that feature constant relative risk aversion (CRRA), so that each household's utility maximization problem is given by

$$\begin{aligned} J(w,t) &= \max_{c_i(t),\phi_i(t)} E_t \left[\int_t^\infty \frac{c_i^{1-\gamma}(s)}{1-\gamma} e^{-\rho s}\, ds \right] \\ \text{s.t.} \quad dw_i(s) &= \left[r w_i(s) + (\alpha - r)\phi_i(s) w_i(s) - c_i(s) + \lambda \right] ds + \sigma \phi_i(s) w_i(s)\, dB_i(s), \end{aligned} \tag{2}$$

where $\rho > 0$ is the discount rate and $\gamma \geq 1$ is the coefficient of relative risk aversion. Because of the symmetry of this setup, all of the N households in the economy have the same functional form for risky-asset demand $\phi_i(t)$ and consumption $c_i(t)$ in equilibrium. The only factors that distinguish these choices among the households are the differential levels of wealth for each household.

Proposition 1. *For all households $i = 1, \ldots, N$, the policy functions $c_i(t)$ and $\phi_i(t)$ are given by*

$$c_i(t) = \left(\frac{\rho - (1-\gamma) r}{\gamma} - \frac{(1-\gamma)(\alpha - r)^2}{2\gamma^2 \sigma^2} \right) \left(w_i(t) + \frac{\lambda}{r} \right), \tag{3}$$

$$\phi_i(t) = \frac{(\alpha - r)\left(w_i(t) + \frac{\lambda}{r} \right)}{w_i(t) \gamma \sigma^2}. \tag{4}$$

The optimal policy functions as given by Proposition 1 are similar to those first characterized by Merton (1969) (the proposition's proof, which is also similar, is in Appendix A). The proposition states that each household invests a quantity of wealth in its individual-specific risky asset that is proportional to the expected instantaneous excess return of that asset, $\alpha - r$, the variance of the risky asset's instantaneous return, σ^2, and the coefficient of relative risk aversion, γ. The intuition behind the roles of these three components of risky-asset demand is standard in macroeconomics and finance. In addition to these well-known channels, we can see that households' ability to risklessly earn labor income throughout their lives also increases their willingness to invest in their risky assets since $\phi_i(t)$ is increasing in λ. The availability of this risk-free labor income makes the risk-free asset somewhat redundant and thus reduces the benefit of investing wealth in safe assets.

The households' optimal choice of consumption is also highly intuitive. The proposition states that each household's consumption is proportional to its total wealth, which is the sum of its physical wealth $w_i(t)$ and its discounted future labor income $\frac{\lambda}{r}$. For each household, the proportion of wealth that is consumed is increasing in the discount rate ρ, the variance of its individual-specific risky asset's instantaneous return σ^2, and the coefficient of relative risk aversion γ, and it is decreasing in the expected instantaneous excess return of the risky asset $\alpha - r$. As with the demand for the risky asset, the intuition behind the roles of these components of consumption is standard.

The setup of this section's model is purposely parsimonious. One advantage of the assumption that households' investment opportunities do not vary over time is that this makes the optimization problem (2) easily solvable. In general, portfolio optimization problems that involve time-varying returns or volatility present difficult challenges.[8] Of course, there is a great deal of empirical evidence that both the returns and volatility of investments vary over time. Our goal in this paper, however, is not to present a complete and intricate model of portfolio choice, but rather to present a simple and stylized model of an economy in which households face idiosyncratic investment risk and there are no redistributive mechanisms. There are many potentially promising and important extensions of this framework, such as the inclusion of richer and more realistic investment returns, transactions costs and other frictions, and behavioral aspects of portfolio choice.

2.2 Household Wealth Dynamics and Redistributive Mechanisms

The next step in our analysis is to characterize the dynamics of wealth for each household in this economy in which there is uninsurable idiosyncratic investment risk. This characterization allows us to describe exactly what a redistributive mechanism is, and then to characterize the dynamic behavior of the equilibrium distribution of wealth in the absence of such mechanisms.

Let $x_i(t)$ be the total wealth of household $i = 1, \ldots, N$ at time t, so that

$$x_i(t) = w_i(t) + \frac{\lambda}{r}. \tag{5}$$

[8] For a detailed discussion of some of these difficulties and the techniques that can be used to address them, see Campbell and Viceira (2002).

Proposition 1 together with the households' budget constraint implies that the dynamics of the total wealth of household i are given by

$$dx_i(t) = \left(\frac{r-\rho}{\gamma} + \frac{(1+\gamma)(\alpha-r)^2}{2\gamma^2\sigma^2} \right) x_i(t)\, dt + \left(\frac{\alpha-r}{\gamma\sigma} \right) x_i(t)\, dB_i(t), \tag{6}$$

so that the total wealth of each household in the economy evolves according to a geometric Brownian motion. The expected instantaneous return of the households' total wealth processes is equal to

$$\Lambda = \frac{r-\rho}{\gamma} + \frac{(1+\gamma)(\alpha-r)^2}{2\gamma^2\sigma^2}, \tag{7}$$

and the instantaneous standard deviation of this return is equal to

$$\Gamma = \frac{\alpha-r}{\gamma\sigma} > 0. \tag{8}$$

According to the definitions of Λ and Γ, equation (6) becomes

$$dx_i(t) = \Lambda x_i(t)\, dt + \Gamma x_i(t)\, dB_i(t), \tag{9}$$

and we can solve this explicitly for the value of each household i's total wealth at time t, which is given by

$$x_i(t) = x_i(0) \exp \left[\left(\Lambda - \frac{1}{2}\Gamma^2 \right) t + \Gamma B_i(t) \right]. \tag{10}$$

It is important to emphasize that equations (9) and (10) imply that all households in the economy have the same expected growth rates of wealth with value $\Lambda - \frac{1}{2}\Gamma^2$. This observation means that there are neither explicit nor implicit redistributive mechanisms present in the economy, since a redistributive mechanism is any factor that proportionally affects wealthy and poor households differently. More precisely, a redistributive mechanism generates an expected growth rate of wealth for wealthy households that is lower than for poor households, which in terms of equations (9) and (10) implies that $\Lambda - \frac{1}{2}\Gamma^2$ is lower for wealthy households. Since each household's expected growth rate of wealth is independent of its current level of wealth, there are no redistributive mechanisms in this setup.

2.3 Dynamics for the Equilibrium Distribution of Wealth

Having characterized the dynamics of total wealth for the N households in the economy, we are ready to analyze the dynamics of the distribution of wealth. In particular, we want to explore the issue of wealth concentration and examine the behavior of the wealthiest households relative to the rest of the population. What happens to the share of the total wealth in the economy that is held by the wealthiest 1% of households? What about the wealthiest 1-5% of households, or the wealthiest single household? These questions are answered by applying some results about the behavior of independent diffusion processes from stochastic calculus.

Before we present these results, however, it is useful to introduce some notation. First, let $x_{\max}(t) = \max\big(x_1(t), \ldots, x_N(t)\big)$, and then let $x(t)$ represent the total wealth in the economy, so that $x(t) = x_1(t) + \cdots + x_N(t) \le N x_{\max}(t)$. Next, let $\theta_i(t)$ be the share of the total wealth in the economy held by the ith household at time t, so that

$$\theta_i(t) = \frac{x_i(t)}{x(t)}, \tag{11}$$

for $i = 1, \ldots, N$. Note that $\theta_i(t) < 1$ for all i and that $\theta_1(t) + \cdots + \theta_N(t) = 1$. Following the definition of $x_{\max}(t)$ from above, let $\theta_{\max}(t)$ be given by

$$\theta_{\max}(t) = \max\big(\theta_1(t), \ldots, \theta_N(t)\big). \tag{12}$$

We are now ready to present the main theorem. The proof is in Appendix A.

Theorem 2. *If households face uninsurable idiosyncratic investment risk, then the share of the economy's total wealth held by the wealthiest single household, $\theta_{\max}$, satisfies*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta_{\max}(t)\, dt = 1, \quad \text{a.s.} \tag{13}$$

Theorem 2 states that the time-averaged share of the total wealth held by the wealthiest single household converges to one, almost surely.[9] In this setting, all of the wealth in the

[9]This theorem can be proved in greater generality. In fact, the same limit result obtains even if the expected growth rate Λ and the instantaneous standard deviation Γ of total wealth as given by equation (9) are time-varying, provided certain conditions are satisfied. See the remarks after the proof of Theorem 2 in Appendix A.

economy eventually concentrates with the wealthiest household. The stark nature of this result is perhaps surprising. Consider: All the households in the economy have identical abilities in terms of their labor incomes and the expected returns of their individual-specific risky assets, and they all have identical patience in terms of their preferences for consumption over time. Furthermore, there is complete equality of opportunity among households, so that the playing field is as level as possible. The implication is that it is luck alone—in the form of high realized investment returns—that generates this extreme divergence.

It is worth noting that the existence of the time average limit in Theorem 2 does not imply that $\theta_{\max}(t)$ itself has the same limit. In fact, the limit of $\theta_{\max}(t)$ as $t \to \infty$ does not exist, as shown by Proposition 3 in Appendix A. Instead, we find that although the time average of $\theta_{\max}(t)$ tends to one, there are recurring "regime changes" in which the wealthiest household in the economy is overtaken by another household. Regimes may last for ever longer periods and involve ever more extreme concentrations of wealth, but they always eventually fall.

The reason that differing investment returns can cause the economy's wealth to concentrate with a single household is that there are no redistributive mechanisms in our setup. In fact, this is the key result of the paper. In the absence of redistribution, uninsurable idiosyncratic investment risk generates an equilibrium wealth distribution that is not stationary and becomes increasingly right-skewed over time. The unmistakeable implication of this result is that those mechanisms that proportionately affect wealthy households and poor households differently play a crucial stabilizing role in the economy. It does not matter whether such redistributive mechanisms are in the form of actions undertaken by individuals or policies implemented by the government. What matters is the presence of such mechanisms, since it is only this presence that prevents an unstable outcome in which a single household eventually holds virtually all of the economy's wealth.

Of course, in most cases some redistributive mechanisms are present in the economy. Settings of this kind, in which households face uninsurable idiosyncratic investment risk and there is redistribution in the form of both limited intergenerational transfers and government tax and fiscal policies, have been the subject of recent work by Benhabib et al. (2011), Benhabib and Zhu (2009), Gabaix (2009), and Zhu (2010). These authors generate equilibrium distributions of wealth that are Pareto and closely match the shape of the U.S. wealth distribution. Similarly, Fernholz (2014) derives related results analytically by extending our continuous-time setting to include redistribution and using recent results from mathematical finance to characterize the stationary Pareto-like equilibrium distribution of wealth.[10] In this paper, however, the goal is to investigate the dynamics of the economy's total wealth in the absence of redistributive mechanisms. This setup is a natural benchmark. Indeed, only by analyzing the behavior of an economy without redistributive mechanisms are we able to uncover the vital stabilizing role that those mechanisms play.

In addition to assuming that there is no redistribution, our setup assumes that households have identical abilities and patience and that there is complete equality of opportunity in the economy. These latter assumptions are clearly not realistic. In the real world, households earn different labor incomes, have different expected returns and variances for their investments, and have different preferences for consumption over time. It is clear, however, that persistent differences in abilities or patience will lead to the same instability as in Theorem 2. Indeed, Becker (1980) showed that different preferences for consumption over time alone leads to full concentration of wealth with a single household.[11] What is surprising is that equality of opportunities, abilities, and patience is not sufficient to ensure stability.

Our results show that, in the absence of perfect insurance and complete markets, the distribution of wealth is unstable without some form of redistribution.[12] While this redistribution can occur in many different ways, one obvious example is redistributive government tax and fiscal policies. Indeed, this section's conclusions suggest an important role for redistributive income and estate taxes as a mechanism that counters the natural tendency of wealth to concentrate at the top. More research that examines the potential for such policies to mitigate income and wealth inequality and compares these benefits with the well-known costs of taxation is warranted.

[10] These mathematical finance results, which are closely related to those we use in this paper, are from Fernholz (2002), Fernholz and Karatzas (2009), and Ichiba et al. (2011).

[11] A simple alternative specification of our model extends this result. Rather than assume that one household's discount rate is larger than other households' discount rates as Becker (1980) does, we could alter our setup so that all households' discount rates follow identical geometric Brownian motion processes (same drift and standard deviation, but independent realizations, just like equation (1) above). In this case, even if all households' realized investment returns are identical, we would still get divergence as in Theorem 2.

[12] Besides the obvious challenges posed by permanently increasing wealth inequality, a more subtle problem involves the link between financial crises and inequality. See Kumhof et al. (2013) and Rajan (2010).

3 Simulations

Given the sharpness of our results, it is helpful to directly observe the dynamics of an economy in which households face idiosyncratic investment risk and there are no redistributive mechanisms. In this section, we present simulations that correspond to several different parameterizations of the previous section's model. This exercise provides us with a number of important observations, including the positive relationship between households' exposure to uninsurable idiosyncratic investment risk and the rate at which the distribution of wealth becomes more right-skewed over time.

We consider a wide range of parameters for the economy. However, in all of this section's parameterizations, we set the total number of households in the economy N equal to 1,000,000 and we assume that all households hold the same amount of wealth at time zero.[13] One implication of the setup of the model is that all households' expected instantaneous returns for total wealth are equal. Recall that equation (9) from the previous section states that

$$dx_i(t) = \Lambda x_i(t)\,dt + \Gamma x_i(t)\,dB_i(t), \tag{14}$$

which implies that for each household $i = 1, \dots, N$, this expected instantaneous return is simply equal to Λ as given by equation (7) above. An important consequence of this fact is that the dynamics of the economy's distribution of wealth over time are completely unaffected by the expected return Λ. Instead, these dynamics are entirely the result of households' different realized rates of investment return as represented by the realizations of the independent Brownian motions in equation (14). As a consequence, then, the distribution of wealth is greatly affected by the standard deviation of these rates of return, which by equation (8), is given by

$$\Gamma = \frac{\alpha - r}{\gamma\sigma}. \tag{15}$$

The endogenous value of Γ measures the extent of households' exposure to uninsurable idiosyncratic investment risk and hence determines the pace at which wealth disperses across households over time.

The fact that only the extent of households' exposure to idiosyncratic risk affects the

[13] In general, we find that changing the number of households in the economy has little effect on the outcome of the simulations. Increasing the number of households makes the plots smoother (at the cost of greater computational intensity), but does not appear to change their overall shape. This is especially true of the plots showing the shares of wealth held by the top 1%, 1-5%, and 5-10%.

dynamics of the distribution of wealth over time simplifies this section's simulation exercise. Rather than depending on all of the model's parameters, these dynamics instead depend only on those parameters that determine the extent of this exposure Γ. According to equation (15) above, then, there are only four parameters that are relevant: the coefficient of relative risk aversion γ, the expected instantaneous return of the risky asset α, the standard deviation of the instantaneous return of the risky asset σ, and the return of the risk-free asset r.[14] Consistent with the standard theory of portfolio choice, the model predicts that households' exposure to idiosyncratic risk increases as α increases and decreases as γ, r, and σ increase.

Since it is only the quantity Γ that affects the dynamics of the distribution of wealth over time, we only need to choose values for those four parameters that affect this quantity. In the benchmark parameterization, we set the coefficient of relative risk aversion $\gamma = 2$ and the risk-free rate of return $r = 0.03$, both values that are consistent with the macroeconomics and finance literature. Parameterizing the returns of households' idiosyncratic investments is more difficult. Both Flavin and Yamashita (2002) and Moskowitz and Vissing-Jorgensen (2002) provide estimates of different components of these idiosyncratic returns, with the former focused on the return of housing and the latter focused on the return of private equity. These analyses led Angeletos (2007) to establish a basic calibration in which the expected investment return is approximately 7% with a standard deviation of 20%. In our benchmark parameterization, we adopt these same values so that $\alpha = 7\%$ and $\sigma = 20\%$, which together with the values for γ and r implies that $\Gamma = 0.1$ in this case.[15]

In Figure 1 below, we plot the shares of the economy's total wealth held by the wealthiest 1% of households (solid black line), the wealthiest 1-5% of households (dotted red line), and the wealthiest 5-10% of households (dashed green line) in a simulation of the previous section's model that corresponds to our benchmark parameterization. The figure shows the dynamics of the distribution of wealth as described by Theorem 2. Starting from a position in which all households hold the same amount of wealth, the economy gradually evolves as wealth concentrates at the top so that within 500 years approximately half of the economy's total wealth is held by the wealthiest 1% of households, and after 1000 years that share increases to nearly 80%. Within 300 years, the economy reaches a position that

[14] There are two other parameters in the model, both of which have no effect on the dynamics of the distribution of wealth over time. Those are the households' discount rate ρ and their labor income λ.

[15] Because it is only the value of Γ that matters for the simulations, the benchmark parameterization can be reinterpreted as any combination of the parameters α, γ, r, and σ that yields $\Gamma = 0.1$. In fact, all of this section's simulations can be reinterpreted in this manner.

resembles the United States in 2001—the wealthiest 1% of households hold 33% of total wealth, while the wealthiest 1-5% and 5-10% of households hold, respectively, 26% and 12% of total wealth.[16]

To examine how the extent of households' exposure to idiosyncratic risk affects the dynamics of the distribution of wealth over time, in Figure 2 we alter the parameters of the model so that $\Gamma = 0.05$ and $\Gamma = 0.2$ and then plot the same shares of wealth as in Figure 1. The shares of wealth over time for $\Gamma = 0.05$ and $\Gamma = 0.2$ are plotted in the top and bottom panels of Figure 2, respectively. The parameterization with $\Gamma = 0.05$ might correspond to an economy in which $\gamma = 4$ (the other parameters are unchanged) and households are significantly more risk-averse than in the benchmark parameterization. Similarly, the parameterization with $\Gamma = 0.2$ might correspond to an economy in which $\sigma = 10\%$ and the standard deviation for the return of investment is lower than in the benchmark parameterization. Of course, any combination of the parameters α, γ, r, and σ that yields the same value of Γ will generate the same predictions for the model.

If $\Gamma = 0.05$ as in the top panel of Figure 2, households endogenously choose less exposure to uninsurable idiosyncratic investment risk and this causes the distribution of wealth to become increasingly right-skewed over time much more slowly than in the benchmark parameterization with $\Gamma = 0.1$ (shown in the middle panel). The economy starts in a position in which all households hold the same amount of wealth and after 1000 years the wealthiest 1% of households hold just under 25% of the economy's total wealth, less than the share of the wealthiest 1-5% of households. Clearly, this outcome is quite different from the benchmark case. Together, the top two panels of Figure 2 show that a decrease in the extent of households' exposure to idiosyncratic risk causes a decrease in the rate at which the economy's total wealth concentrates at the top. Given this observation, we should expect that an increase in this exposure will cause an increase in the pace of wealth concentration over time.

This conjecture is examined in the bottom panel of Figure 2. In this alternate parameterization $\Gamma = 0.2$, so households endogenously choose more exposure to idiosyncratic investment risk which causes the distribution of wealth to become increasingly right-skewed over time much more quickly than in the benchmark parameterization. Starting from a position in which all households hold the same amount of wealth, the economy evolves so that

[16]These data are from Wolff (2012).

within 600 years practically all of the economy's total wealth is held by the wealthiest 1% of households, with none held by the remaining 99% of households. Furthermore, we can see from this bottom panel that within one century the model predicts that more than one third of the economy's total wealth will be held by the wealthiest 1% of households. This pace of increasing right-skewness over time far exceeds what is observed in the top two panels of Figure 2.

Figure 2 demonstrates one of the most interesting and significant properties of the model we present in this paper. It is the extent of households' exposure to uninsurable idiosyncratic investment risk, measured by the quantity Γ, that determines the pace at which the economy's total wealth concentrates at the top. Consider the evolution of the distribution of wealth over a period of 200 years. Our simulations show that as the value of Γ fluctuates between 0.05 and 0.2, the share of the economy's total wealth held by the wealthiest 1% of households after 200 years fluctuates between 5% and 70%. These results indicate that the rate at which the convergence described by Theorem 2 occurs varies significantly depending upon the value of Γ.

Importantly, the extent of households' exposure to idiosyncratic risk as measured by Γ is endogenous in our model. This value is a function of households' risk aversion γ, the expected excess return of the risky asset relative to the risk-free asset $\alpha - r$, and the standard deviation of the return of this risky asset σ. Consistent with the standard theory of portfolio choice, the model predicts that households' exposure to idiosyncratic risk increases in the excess return and decreases in risk aversion and the standard deviation. Of course, the existence of uninsurable risk is a consequence of market incompleteness, and so these results point to a close relationship between the extent of market incompleteness and inequality. This conclusion, which is broadly consistent with the conclusions of Benhabib et al. (2011) and Zhu (2010), gives a renewed significance to theoretical and empirical studies of the causes and consequences of incomplete markets. Indeed, if market incompleteness and idiosyncratic investment risk are major causes of income inequality, then the fact that markets are incomplete is much more than just an interesting and potentially surprising observation. In fact, such incompleteness may be a major cause of increasing income and wealth inequality as individuals who repeatedly experience high realized rates of return on their investments accumulate ever-larger shares of wealth. More research is clearly warranted in this direction.

The next step in this section's simulation exercise is to analyze how the Gini coefficient of our economy evolves over time. By Theorem 2, we know that the Gini coefficient will eventually converge to 1.0, but we do not know how fast such a convergence will occur and how this rate of convergence is influenced by the parameters of the model. In Figure 3, we plot the evolution of the Gini coefficient over time for the benchmark parameterization in which $\Gamma = 0.1$ (the solid black line). Starting from a position in which all households hold the same amount of wealth and the Gini coefficient is equal to 0, the economy evolves so that within 400 years the Gini coefficient is just above 0.8 and approximately equal to the true value in both the United States and the world in 2000. Figure 3 also demonstrates that the Gini coefficient grows rapidly at first but eventually slows down as it approaches a value of one. According to the figure, the value of the coefficient jumps from 0 to 0.5 within 100 years but then only increases up to 0.9 over the next 500 years.

The positive relationship between the extent of households' exposure to idiosyncratic risk and the rate at which wealth concentrates at the top as shown by Figure 2 suggests that a similar pattern should emerge in the dynamics of the economy's Gini coefficient over time. To see that this is indeed the case, in Figure 3 we also include plots of the evolution of the Gini coefficient of the economy over time for the low-exposure ($\Gamma = 0.05$, represented by the dotted red line) and high-exposure ($\Gamma = 0.2$, represented by the dashed green line) parameterizations described above. According to the figure, the low-exposure parameterization of our model generates a Gini coefficient approximately equal to 0.5 after 400 years. Furthermore, the coefficient in this case does not even reach the previously observed value of 0.8 within 1000 years. The high-exposure parameterization of our model, in contrast, generates a Gini coefficient around 0.8 in less than 100 years and close to 1.0 within only 200 years. This represents a very rapid concentration of wealth at the top. As before, these results highlight just how important the extent of households' exposure to idiosyncratic investment risk is when determining both how rapidly wealth inequality increases over time and how rapidly the convergence described by Theorem 2 occurs.

Figures 1-3 are all roughly consistent with the predictions of Section 2 and Theorem 2. However, the theorem states a result that is substantially stronger than simply an increasing share of wealth held by the wealthiest 1% of households or an increasing Gini coefficient over time. The theorem states that a single household will eventually hold virtually all of the economy's wealth. This extreme scenario represents the most unequal distribution of wealth in the economy that is possible. In Figure 4 below, we plot the share of total wealth held by the single wealthiest household in the economy for the high-exposure parameterization in which $\Gamma = 0.2$. The extreme concentration of wealth with a single household is an event that takes a long time to occur, and so we plot the evolution of this household's share of wealth over a 10,000 year period.

From the figure, we see that in a little over 1000 years, practically all of the economy's wealth is held by a single household, but this outcome lasts less than a few hundred years as other households in the economy realize high rates of return on their investments and are able to increase their shares of the economy's total wealth. Figure 4 demonstrates that changes in the share of wealth held by the wealthiest household in the economy are frequent and large. Indeed, approximately 500 years after holding all of the economy's total wealth, the wealthiest household in the economy holds less than 20% of this total wealth. Such large fluctuations are common throughout Figure 4, although virtually all of the economy's wealth is held by a single household for nearly all of a period of roughly 3000 years. This extended period of complete wealth concentration at the top is consistent with the figure's overall trend of increasing concentration, volatile as it is.[17] Notably, the substantial variation that is displayed in Figure 4 contrasts with the smooth and stable behavior of Figures 1-3. This demonstrates that the total and permanent concentration of the economy's wealth in our setup only occurs after a long period of sizable fluctuations at the very top of the wealth distribution. Indeed, while there is much variation in the share of wealth held by the wealthiest household in the economy, there is practically no variation in the share of wealth held by the wealthiest 1% of households (which corresponds to the wealthiest 10,000 households in the economy).

4 Conclusion

In this paper, we have theoretically examined the dynamics of the distribution of wealth in an economy in which infinitely-lived households face idiosyncratic investment risk and make optimal decisions about how much to consume and how to invest. A central feature of our setup is that neither explicit nor implicit redistributive mechanisms are present in the economy. Furthermore, all households are assumed to be equally patient and have identical abilities. In this setting, we show that the equilibrium distribution of wealth is not stationary

[17] The high degree of volatility that is present in Figure 4 is readily observed after a number of simulations.

and prove that it becomes increasingly right-skewed over time and gradually tends to a limit in which wealth is fully concentrated at the top. This extreme divergence is driven entirely by luck in the form of high realized investment returns.

We consider a number of different parameterizations of our model and present the corresponding simulations. This exercise reveals a link between the extent of households' exposure to idiosyncratic investment risk and the pace at which the distribution of wealth becomes increasingly right-skewed over time. The greater the households' exposure to idiosyncratic risk, the faster the economy's total wealth accumulates with the wealthiest single household. This exposure is endogenous in our setup. In particular, it increases as the expected excess return of the households' risky investments increases and decreases as both the variance of these returns and the households' risk-aversion increase. Given the important role of incomplete markets and uninsurable risk in our conclusions, more research in this direction—both theoretically, exploring the endogenous causes of incomplete markets, and empirically, examining which risks do not become insured—may yield significant insights.

The main implication of our analysis is that in the absence of complete markets and perfect insurance, redistributive mechanisms play a crucial stabilizing role in the economy. What matters is not whether such redistributive mechanisms come in the form of actions undertaken by individuals or policies implemented by the government, but rather whether or not a sufficient level of redistribution is achieved. This conclusion suggests that there is a potentially important role for redistributive government tax and fiscal polices in an economy. Surely, further examining the stabilizing role of such policies and contrasting this benefit with the traditional distortionary costs of taxation should be an objective for future research.

A Proofs

This appendix presents the proofs of Propositions 1 and 3 and Theorem 2.

Proof of Proposition 1. Under suitable regularity conditions, Itô's Lemma implies that the Hamilton-Jacobi-Bellman equation for household i's maximization problem is given by

$$\begin{aligned} 0 &= \max_{c_i(t),\phi_i(t)} \left\{ e^{-\rho t} \frac{c_i^{1-\gamma}(t)}{1-\gamma} + J_w(w,t)[rw_i(t) + (\alpha - r)\phi_i(t)w_i(t) - c_i(t) + \lambda] \right. \\ & \qquad \left. + J_t(w,t) + \frac{1}{2} J_{ww}(w,t)\phi_i^2(t)\sigma^2 w_i^2(t) \right\}, \end{aligned}$$

where $J_w(w,t)$ and $J_t(w,t)$ denote respectively the partial derivatives of the value function with respect to wealth w and time t. The first-order conditions for this maximization problem are therefore

$$c_i^{-\gamma}(t) = e^{\rho t} J_w(w,t), \tag{16}$$

$$J_w(w,t)(\alpha - r)w_i(t) = -J_{ww}(w,t)\phi_i(t)\sigma^2 w_i^2(t). \tag{17}$$

The next step is to guess and verify the form of the value function $J(w,t)$. We guess that

$$J(w,t) = e^{-\rho t} \frac{k}{1-\gamma} \left(w_i(t) + \frac{\lambda}{r} \right)^{1-\gamma}, \tag{18}$$

where k is a positive constant, so that the first-order conditions (16) and (17) imply that

$$c_i(t) = k^{-\frac{1}{\gamma}} \left(w_i(t) + \frac{\lambda}{r} \right), \tag{19}$$

$$\phi_i(t) = \frac{(\alpha - r)\left(w_i(t) + \frac{\lambda}{r} \right)}{w_i(t)\gamma\sigma^2}. \tag{20}$$

Note that the expression for the optimal holdings of the risky asset $\phi_i(t)$ given by equation (20) confirms equation (4) from Proposition 1.

The last step of this proof is to solve for the positive constant k that is both part of the value function and the optimal level of consumption and to confirm that the value function from (18) does indeed satisfy the Hamilton-Jacobi-Bellman equation from above. This is accomplished by substituting the optimal levels of consumption and investment as given by equations (19) and (20) into the Hamilton-Jacobi-Bellman equation. This yields

$$0 \;=\; e^{-\rho t}\frac{k^{1-\frac{1}{\gamma}}}{1-\gamma}\left(w_i(t)+\frac{\lambda}{r}\right)^{1-\gamma} + \; e^{-\rho t}k\left(w_i(t)+\frac{\lambda}{r}\right)^{1-\gamma}\left[r+\frac{(\alpha-r)^2}{\gamma\sigma^2}-k^{-\frac{1}{\gamma}}\right]$$

$$-\;\frac{1}{2}e^{-\rho t}\gamma k\left(w_i(t)+\frac{\lambda}{r}\right)^{1-\gamma}\frac{(\alpha-r)^2}{\gamma^2\sigma^2}-\rho e^{-\rho t}\frac{k}{1-\gamma}\left(w_i(t)+\frac{\lambda}{r}\right)^{1-\gamma},$$

which, after simplifying, implies that the equality holds with

$$k^{-\frac{1}{\gamma}} = \frac{\rho-(1-\gamma)r}{\gamma} - \frac{(1-\gamma)(\alpha-r)^2}{2\gamma^2\sigma^2}.$$

If we substitute this expression into equation (19) above, then we get that optimal consumption $c_i(t)$ is given by equation (3) from Proposition 1, completing the proof. □

Proof of Theorem 2. According to Proposition 1 and equation (9), the total wealth of household i, denoted by x_i, evolves according to

$$dx_i(t) = \Lambda x_i(t)\,dt + \Gamma x_i(t)\,dB_i(t), \tag{21}$$

which, by Itô's Lemma, becomes

$$d\log x_i(t) = \left(\Lambda - \frac{1}{2}\Gamma^2\right)\,dt + \Gamma\,dB_i(t). \tag{22}$$

Without loss of generality, we can rescale the above stochastic process for log-wealth so that $\Lambda - \frac{1}{2}\Gamma^2 = 0$, and we can make a time change so that $\Gamma = 1$. With these changes, equation (22) reduces to

$$d\log x_i(t) = dB_i(t), \tag{23}$$

and (21) can be written as

$$\frac{dx_i(t)}{x_i(t)} = \frac{1}{2}\,dt + dB_i(t). \tag{24}$$

If we assume initial values $x_i(0) > 0$, for $i = 1,\ldots,N$, then (23) results in

$$\log x_i(t) = B_i(t) + \log x_i(0),$$

or

$$x_i(t) = x_i(0)\exp(B_i(t)).$$

Let us now consider the dynamics of the processes x_i. For $1 \leq i \leq N$, the strong law of large numbers for Brownian motion (see Karatzas and Shreve, 1991, Problem 2.9.3) implies that

$$\lim_{t\to\infty} \frac{\log x_i(t)}{t} = \lim_{t\to\infty} \frac{B_i(t)}{t} = 0, \quad \text{a.s.},$$

so the same limit holds for $\log x_{\max}(t)$. Since

$$\log x_1(t) \leq \log x(t) = \log\big(x_1(t) + \cdots + x_N(t)\big) \leq \log N + \log x_{\max}(t),$$

we also have

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = 0, \quad \text{a.s.} \tag{25}$$

By Itô's Lemma we have

$$dx(t) = d\exp(\log x(t)) = x(t)\,d\log x(t) + \frac{x(t)}{2} d\langle \log x\rangle_t, \quad \text{a.s.},$$

where $\langle \log x\rangle_t$ denotes the quadratic variation of $\log x$ up to time t, and this is equivalent to

$$\frac{dx(t)}{x(t)} = d\log x(t) + \frac{1}{2}\, d\langle \log x\rangle_t, \quad \text{a.s.} \tag{26}$$

Since

$$dx(t) = \sum_{i=1}^N dx_i(t),$$

we have

$$\begin{aligned} \frac{dx(t)}{x(t)} &= \sum_{i=1}^N \frac{x_i(t)}{x(t)} \frac{dx_i(t)}{x_i(t)} \\ &= \sum_{i=1}^N \theta_i(t) \frac{dx_i(t)}{x_i(t)}, \end{aligned} \tag{27}$$

with $\theta_i(t) = x_i(t)/x(t)$ as in (11). From (24) and (27), it follows that

$$\frac{dx(t)}{x(t)} = \sum_{i=1}^{N} \theta_i(t) dB_i(t) + \frac{1}{2}\, dt, \tag{28}$$

so (26) implies that

$$d \log x(t) = \sum_{i=1}^{N} \theta_i(t) dB_i(t) + \frac{1}{2}\, dt - \frac{1}{2}\, d\langle \log x\rangle_t, \quad \text{a.s.} \tag{29}$$

Since the B_i are independent, we have

$$d\langle \log x\rangle_t = \sum_{i=1}^{N} \theta_i^2(t)\, dt, \quad \text{a.s.,}$$

which means that (29) can be expressed as

$$d \log x(t) = \sum_{i=1}^{N} \theta_i(t) dB_i(t) + \frac{1}{2}\Big(1 - \sum_{i=1}^{N} \theta_i^2(t)\Big) dt, \quad \text{a.s.} \tag{30}$$

All the proportions θ_i are adapted and bounded, so the process

$$M(t) = \int_0^t \sum_{i=1}^{N} \theta_i(t)\, dB_i(t)$$

is a local martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \sum_{i=1}^{N} \theta_i^2(s)\, ds, \quad \text{a.s.,}$$

and with

$$\frac{1}{N}\, t \leq \langle M\rangle_t \leq t, \quad \text{a.s.} \tag{31}$$

By the time-change theorem for martingales (see Karatzas and Shreve, 1991, Theorem 3.4.6), there exists a Brownian motion B' such that

$$M(t) = B'(\langle M\rangle_t), \quad \text{a.s.,}$$

for $t \in [0,\infty)$. We now have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{i=1}^N \theta_i(t) dB_i(t)\, dt = \lim_{T\to\infty} \frac{M(T)}{T} = \lim_{T\to\infty} \frac{B'(\langle M\rangle_T)}{\langle M\rangle_T}\frac{\langle M\rangle_T}{T} = 0 \quad \text{a.s.},$$

by (31) and the strong law of large numbers for Brownian motion. Hence, from (25) and (30) we can conclude that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(1 - \sum_{i=1}^N \theta_i^2(t)\Big) dt = 0, \quad \text{a.s.} \tag{32}$$

Now,

$$1 - \sum_{i=1}^N \theta_i^2(t) = \sum_{i=1}^N \big(\theta_i(t) - \theta_i^2(t)\big) = \sum_{i=1}^N \theta_i(t)\big(1-\theta_i(t)\big) \geq 1 - \theta_{\max}(t) \geq 0,$$

so (32) implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(1 - \theta_{\max}(t)\big) dt = 0, \quad \text{a.s.},$$

which is equivalent to equation (13). □

Remark. Theorem 2 can be proved in greater generality. Suppose that the total wealth processes for the N households in the economy are a system of continuous semimartingales $x_1,\ldots,x_N$ such that for $i = 1,\ldots,N$,

$$d\log x_i(t) = \beta(t)\, dt + \sum_{\nu=1}^d \xi_{i\nu}(t)\, dB_\nu(t),$$

where $(B_1,\ldots,B_d)$ is a d-dimensional Brownian motion with $d \geq N$, and β and $\xi_{i\nu}$ are measurable and adapted to the Brownian filtration, with β locally L^1 and $\xi_{i\nu}$ locally L^2. For $1 \leq i,j \leq N$, we define the *covariance processes* by

$$s_{ij}(t) \equiv \sum_{\nu=1}^d \xi_{i\nu}(t)\xi_{j\nu}(t),$$

and let us assume that the s_{ij} satisfy

$$\lim_{t\to\infty} \frac{\log\log t}{t} s_{ij}(t) = 0, \quad \text{a.s.},$$

for all i and j, and that the covariance matrix $(s_{ij}(\cdot))_{\{1\le i,j\le N\}}$ is *strongly non-degenerate*, i.e., it is positive definite with eigenvalues uniformly bounded away from zero, almost surely. Under these conditions,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta_{\max}(t)\, dt = 1,$$

as in equation (13) from Theorem 2. A proof of this result can be found in Fernholz and Karatzas (2009), Section 5. □

Remark. Another generalization of Theorem 2 is possible if the total wealth processes for the N households in the economy are a system of continuous semimartingales $x_1, \ldots, x_N$ such that for $i = 1, \ldots, N$,

$$d \log x_i(t) = z_i(t)\, dt + dB_i(t),$$

where z_i is an adapted process that satisfies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta_i(t) z_i(t)\, dt \ge 0, \quad \text{a.s.} \tag{33}$$

These dynamics for the total wealth of the N households corresponds to an extension of the benchmark model in which the households have different time-varying abilities, as measured by the average growth rate of total wealth. The condition (33) states that on average over time those households with greater ability hold more wealth than those households with less ability, a restriction that seems entirely reasonable. Under the stated conditions, we once again have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta_{\max}(t)\, dt = 1, \quad \text{a.s.},$$

as in Theorem 2. The proof of this result is almost identical to the proof of the theorem itself. The key difference is that equation (28) now becomes

$$\frac{dx(t)}{x(t)} = \sum_{i=1}^N \theta_i(t) dB_i(t) + \frac{1}{2}\, dt + \sum_{i=1}^N \theta_i(t) z_i(t)\, dt,$$

which by the same arguments as in the proof of the theorem implies that

$$d \log x(t) = \sum_{i=1}^{N} \theta_i(t) dB_i(t) + \frac{1}{2}\left(1 - \sum_{i=1}^{N} \theta_i^2(t)\right) dt + \sum_{i=1}^{N} \theta_i(t) z_i(t)\, dt, \quad \text{a.s.} \tag{34}$$

Following the proof of the theorem once again, we have by equation (34) that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{2}\left(1 - \sum_{i=1}^{N} \theta_i^2(t)\right) dt = - \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{i=1}^{N} \theta_i(t) z_i(t)\, dt \leq 0, \quad \text{a.s.,} \tag{35}$$

which is stronger than what is needed to reach the desired conclusion. □

Proposition 3. *The maximum proportion process $\theta_{\max}$ satisfies*

$$\limsup_{t\to\infty} \theta_{\max}(t) = 1, \quad \text{a.s.,} \tag{36}$$

and

$$\liminf_{t\to\infty} \theta_{\max}(t) \leq \frac{1}{2}, \quad \text{a.s.} \tag{37}$$

Proof. The limit (36) follows immediately from Theorem 2.2 and the fact that $\theta_{\max}(t) < 1$ for all $t \geq 0$.

Since we have the limit in (36), there exist almost surely arbitrarily large $t_0 > 0$ such that $\theta_{\max}(t_0) > 1/2$. Suppose that $\theta_{\max}(t_0) = \theta_i(t_0)$, so $x_{\max}(t_0) = x_i(t_0)$. For any $j \neq i$, $1 \leq j \leq N$, the process v defined for $t \geq 0$ by

$$v(t) = \frac{1}{\sqrt{2\Gamma}}\big(\log x_i(t) - \log x_j(t)\big)$$

is a Brownian motion, so there almost surely exists a $t_1 > t_0$ such that $v(t_1) = 0$. In this case $x_i(t_1) = x_j(t_1)$, so $\theta_i(t_1) < 1/2$. The process θ_i is almost surely continuous, so there exists a smallest $T > t_0$ such that $\theta_i(T) = 1/2$. If $\theta_i(t) > 1/2$, then $\theta_i(t) = \theta_{\max}(t)$, so T is also the smallest $T > t_0$ such that $\theta_{\max}(T) = 1/2$. Since t_0 was arbitrarily large, and $T > t_0$, (37) follows. □

B Discussion: Intergenerational Transfers and Redistributive Mechanisms

In this appendix, we examine the implications of different assumptions about intergenerational transfers for our results about the instability of the distribution of wealth in the presence of idiosyncratic investment risk and no redistributive mechanisms. In particular, we alter the main model of Section 2 so that households have finite lifespans and "joy of giving" bequest motives in a manner that is similar to Zhu (2010). The main result is that intergenerational transfers in this setting are an implicit redistributive mechanism only if households receive positive riskless labor incomes and the intensity of their bequest motives is sufficiently low. Furthermore, we show that the size of intergenerational transfers alone is essentially irrelevant to the stability of the distribution of wealth. Instead, stability relies on a subtle interaction between positive discounted labor incomes and limited growth of household wealth.

Consider an economy that is identical to the setup from Section 2, except that now we assume that the economy is populated by $N \in \mathrm{N}$ dynastic households. Each household lives only for a time of length $T < \infty$, and then at the end of its life passes on any remaining wealth to one child who starts its life at that instant in time. Just like their preferences for consumption, households have a preference for leaving bequests to their children that features constant relative risk aversion (CRRA). The utility maximization problem for each household $i = 1, \ldots, N$ born at time $s \in \{0, T, 2T, \ldots\}$ is given by

$$J(w,s,t) = \max_{c_i(s,t),\phi_i(s,t)} E_t\left[\int_t^T \frac{c_i^{1-\gamma}(s,v)}{1-\gamma} e^{-\rho(v-t)}\,dv + \chi \frac{w_i^{1-\gamma}(s,T)}{1-\gamma} e^{-\rho(T-t)}\right]$$

$$\text{s.t.} \quad dw_i(s,v) = [rw_i(s,v) + (\alpha - r)\phi_i(s,v)w_i(s,v) - c_i(s,v) + \lambda]\,dv + \sigma\phi_i(s,v)w_i(s,v)\,dB_i(v),$$

where $0 \le t < T$ and the intensity of households' bequest motives is parameterized by $\chi > 0$. All other parameters are as in Section 2. Note that there are no explicit redistributive government tax or fiscal policies in this altered setup. Before describing the households' optimal risky-asset demand $\phi_i(s,t)$ and consumption $c_i(s,t)$, it is useful to denote a household's time-discounted future lifetime labor income by

$$h(t) = \frac{1 - e^{-r(T-t)}}{r}\lambda. \tag{38}$$

Proposition 4. *In this altered setup, for all households $i = 1, \dots, N$, the policy functions $c_i(s,t)$ and $\phi_i(s,t)$ are given by*

$$c_i(s,t) = \left(\frac{e^{\eta(T-t)} - 1}{\eta} + \chi^{\frac{1}{\gamma}} e^{\eta(T-t)} \right)^{-1} (w_i(s,t) + h(t)), \tag{39}$$

$$\phi_i(s,t) = \frac{(\alpha - r)(w_i(s,t) + h(t))}{w_i(s,t)\gamma\sigma^2}, \tag{40}$$

where

$$\eta = \frac{(1-\gamma)r - \rho}{\gamma} + \frac{(1-\gamma)(\alpha - r)^2}{2\gamma^2\sigma^2}. \tag{41}$$

Proof. Under suitable regularity conditions, Itô's Lemma implies that the Hamilton-Jacobi-Bellman equation for household i's maximization problem is given by

$$0 = \max_{c_i(s,t), \phi_i(s,t)} \left\{ e^{-\rho t} \frac{c_i^{1-\gamma}(s,t)}{1-\gamma} + J_w(w,s,t) \left[r w_i(s,t) + (\alpha - r)\phi_i(s,t) w_i(s,t) - c_i(s,t) + \lambda \right] \right.$$
$$\left. + J_t(w,s,t) + \frac{1}{2} J_{ww}(w,s,t) \phi_i^2(s,t) \sigma^2 w_i^2(s,t) \right\},$$

where $J_w(w,s,t)$ and $J_t(w,s,t)$ denote respectively the partial derivatives of the value function with respect to wealth w and time t. The first-order conditions for this maximization problem are therefore

$$c^{-\gamma}(s,t) = e^{\rho t} J_w(w,s,t), \tag{42}$$

$$J_w(w,s,t)(\alpha - r) w_i(s,t) = -J_{ww}(w,s,t)\phi_i(s,t)\sigma^2 w_i^2(s,t). \tag{43}$$

The next step is to determine the form of the value function $J(w,s,t)$. Suppose that

$$J(w,s,t) = e^{-\rho t} a(t) \frac{(w_i(s,t) + h(t))^{1-\gamma}}{1-\gamma}, \tag{44}$$

where $a(t)$ is a positive function of t. The first-order conditions (42) and (43) therefore imply that

$$c_i(s,t) = a^{-\frac{1}{\gamma}}(t) (w_i(s,t) + h(t)), \tag{45}$$

$$\phi_i(s,t) = \frac{(\alpha - r)(w_i(s,t) + h(t))}{w_i(s,t)\gamma\sigma^2}. \tag{46}$$

Note that the expression for the optimal holdings of the risky asset $\phi_i(s,t)$ given by equation (46) confirms equation (40) from above.

The last step of this proof is to solve for the function $a(t)$ that is both part of the value function and the optimal level of consumption and to confirm that the value function from (44) does indeed satisfy the Hamilton-Jacobi-Bellman equation from above. This is accomplished by substituting the optimal levels of consumption and investment as given by equations (45) and (46) into the Hamilton-Jacobi-Bellman equation. This yields

$$0 = \frac{a^{1-\frac{1}{\gamma}}(t)}{1-\gamma}\left(w_i(s,t)+h(t)\right)^{1-\gamma} + a(t)\left(w_i(s,t)+h(t)\right)^{1-\gamma}\left[r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a^{-\frac{1}{\gamma}}(t)\right]$$
$$+ \frac{\dot{a}(t)}{1-\gamma}\left(w_i(s,t)+h(t)\right)^{1-\gamma} - \frac{1}{2}a(t)\frac{(\alpha-r)^2}{\gamma\sigma^2}\left(w_i(s,t)+h(t)\right)^{1-\gamma} - \rho\frac{a(t)}{1-\gamma}\left(w_i(s,t)+h(t)\right)^{1-\gamma},$$

which, after simplifying, implies that

$$0 = \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)\dot{a}(t) + \eta a^{\frac{1}{\gamma}}(t) + 1, \tag{47}$$

where η is given by equation (41) above. We can use the boundary condition $a(T) = \chi$ to solve equation (47). This yields the expression

$$a(t) = \left(\frac{e^{\eta(T-t)}-1}{\eta} + \chi^{\frac{1}{\gamma}}e^{\eta(T-t)}\right)^{\gamma},$$

which after substituting into equation (45) above, confirms that optimal consumption $c_i(s,t)$ is given by equation (39). $\square$

If, as in equation 5 above, we define $x_i(s,t)$ as the total wealth of household i born at time $s \in \{0, T, 2T, \ldots\}$, where $0 \leq t < T$ then we have $x_i(s,t) = w_i(s,t) + h(t)$. According to Proposition 4, the dynamics of the total wealth of household i are given by

$$dx_i(s,t) = \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - \left[\frac{e^{\eta(T-t)}-1}{\eta} + \chi^{\frac{1}{\gamma}}e^{\eta(T-t)}\right]^{-1}\right)x_i(s,t)\,dt$$
$$+ \left(\frac{\alpha-r}{\gamma\sigma}\right)x_i(s,t)\,dB_i(s+t).$$

After a few manipulations, it can be shown that these dynamics imply that

$$
\begin{aligned}
x_i(s,t) = x_i(s,0) &\left(\frac{\left(1+\eta\chi^{\frac{1}{\gamma}}\right) e^{\eta(T-t)} - 1}{\left(1+\eta\chi^{\frac{1}{\gamma}}\right) e^{\eta T} - 1} \right) \\
&\exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) t + \left(\frac{\alpha-r}{\gamma\sigma} \right) \left(B_i(s+t) - B_i(s)\right) \right].
\end{aligned} \tag{48}
$$

Note that $x_i(s,0) = x_i(s-T,T) + h(0)$ by definition, so it follows from equation (48) that

$$
\begin{aligned}
x_i(s,0) = x_i(s-T,0) &\left(\frac{\eta\chi^{\frac{1}{\gamma}}}{\left(1+\eta\chi^{\frac{1}{\gamma}}\right) e^{\eta T} - 1} \right) \\
&\exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) T + \left(\frac{\alpha-r}{\gamma\sigma} \right) \left(B_i(s) - B_i(s-T)\right) \right] + h(0).
\end{aligned} \tag{49}
$$

For all $i = 1, \ldots, N$, let $x_i(s+t)$ denote the total wealth of dynastic household i at time $s+t$, where $s \in \{0, T, 2T, \ldots\}$ and $0 \le t < T$. Combining equations (48) and (49) from above, it is not difficult to show that

$$
\begin{aligned}
x_i(s+t) = w_i(0)\psi^{s/T}(T)\psi(t) &\exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) (s+t) + \left(\frac{\alpha-r}{\gamma\sigma} \right) B_i(s+t) \right] \\
&+ h(0)\psi(t) \exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) t + \left(\frac{\alpha-r}{\gamma\sigma} \right) \left(B_i(s+t) - B_i(s)\right) \right] \\
&\Bigg(1 + \psi(T) \exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) T + \left(\frac{\alpha-r}{\gamma\sigma} \right) \left(B_i(s) - B_i(s-T)\right) \right] \\
&+ \psi^2(T) \exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) 2T + \left(\frac{\alpha-r}{\gamma\sigma} \right) \left(B_i(s) - B_i(s-2T)\right) \right] \\
&+ \cdots + \psi^{s/T}(T) \exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} \right) s + \left(\frac{\alpha-r}{\gamma\sigma} \right) B_i(s) \right] \Bigg),
\end{aligned} \tag{50}
$$

where

$$
\psi(t) = \frac{\left(1+\eta\chi^{\frac{1}{\gamma}}\right) e^{\eta(T-t)} - 1}{\left(1+\eta\chi^{\frac{1}{\gamma}}\right) e^{\eta T} - 1}. \tag{51}
$$

Proposition 5. *There exists a finite threshold $\hat{\chi} > 0$ such that if either $\chi \ge \hat{\chi}$ or $h(0) = 0$, then we cannot conclude that the distribution of wealth in this economy with overlapping generations and intergenerational transfers is stationary.*

Proof. Suppose first that $h(0) = 0$. In this case, equation (50) implies that the total wealth of dynastic household i at time $s + t$ is given by

$$x_i(s+t) = x_i(0)\psi^{s/T}(T)\psi(t)\exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)(s+t) + \left(\frac{\alpha-r}{\gamma\sigma}\right)B_i(s+t)\right],$$

since $x_i(0) = w_i(0)$ whenever $h(0) = 0$. Of course, this implies that the dynamics of the process x_i are given by

$$\begin{aligned} dx_i(s+t) = &\left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - \left[\frac{e^{\eta(T-t)}-1}{\eta} + \chi^{\frac{1}{\gamma}}e^{\eta(T-t)}\right]^{-1}\right)x_i(s+t)\,dt \\ &+ \left(\frac{\alpha-r}{\gamma\sigma}\right)x_i(s+t)\,dB_i(s+t), \end{aligned}$$

where $s \in \{0, T, 2T, \ldots\}$ and $0 \leq t < T$. The two remarks in Appendix A thus imply that the distribution of wealth is not stationary in this case, but instead becomes increasingly concentrated over time as in equation (13) from Theorem 2 above.

The next step is to consider the dynamics of wealth over time in the limit as $\chi \to \infty$. In order to show that these dynamics lead to a non-stationary distribution of wealth, it is necessary to examine the wealth accumulation process across generations. For all $i = 1, \ldots, N$ and $n \geq 1$, let $\hat{x}_{in}$ be the wealth of the newborn n-th generation household i. Note that $\hat{x}_{in} = x_i(s+0)$ and $\hat{x}_{in+1} = x_i(s+T)$, where $s = (n-1)T$. It follows, then, by equations (50) and (51), that

$$\hat{x}_{in+1} = \phi_n\hat{x}_{in} + h(0), \tag{52}$$

where ϕ_n is a sequence of random variables such that for all $n \geq 1$,

$$\begin{aligned} \phi_n = &\left(\frac{\eta\chi^{\frac{1}{\gamma}}}{\left(1+\eta\chi^{\frac{1}{\gamma}}\right)e^{\eta T}-1}\right) \\ &\exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T + \left(\frac{\alpha-r}{\gamma\sigma}\right)(B_i(nT) - B_i((n-1)T))\right]. \end{aligned} \tag{53}$$

Because of the independence of Brownian motion increments, it follows that the random sequence ϕ_n is i.i.d. In this case, both Benhabib et al. (2011) and Zhu (2010) have shown, using results from Goldie (1991) and Kesten (1973), that the equilibrium distribution of wealth (as characterized by each generation of newborn households' endowments of wealth

$\hat{x}_{in}$) is stationary and Pareto, provided that there exists some $\mu > 1$ such that $E\phi_n^{\mu} = 1$. According to equation (53), as the intensity of the households' bequest motives $\chi \to \infty$, we have

$$\phi_n \to \exp\left[\left(\frac{r-\rho}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2} - \eta\right)T + \left(\frac{\alpha-r}{\gamma\sigma}\right)(B_i(nT) - B_i((n-1)T))\right],$$

which, by equation (41), implies that

$$\lim_{\chi\to\infty} \phi_n = \exp\left[\left(r + \frac{(2\gamma-1)(\alpha-r)^2}{2\gamma^2\sigma^2}\right)T + \left(\frac{\alpha-r}{\gamma\sigma}\right)(B_i(nT) - B_i((n-1)T))\right].$$

Using the formula for the expected value of a lognormal random variable, it follows that

$$\lim_{\chi\to\infty} E\phi_n = \exp\left[\left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2}\right)T\right] > 1.$$

By Hölder's inequality, we know that $E\phi_n > 1$ implies that $E\phi_n^{\mu} > 1$ for all $\mu > 1$. Thus, there exists a finite threshold $\hat{\chi} > 0$ such that whenever $\chi \geq \hat{\chi}$, $E\phi_n^{\mu} > 1$ for all $\mu > 1$. In this case, we can no longer conclude that there is an equilibrium stationary Pareto distribution of wealth in this economy with intergenerational transfers. □

According to Proposition 5, a stable distribution of wealth requires both limited intergenerational transfers and positive human wealth at the start of households' lives. These two factors interact to create an implicit redistributive mechanism that elevates the expected growth rate of wealth for poor households above that for wealthy households.[18] This occurs because the discounted labor incomes that all households acquire at the start of their lives, denoted by $h(0)$, provide proportionally larger increases in wealth for poorer households.[19] A redistributive mechanism is defined as any process that proportionally affects wealthy and poor households differently, so these positive shocks to human wealth across generations are clearly redistributive.

The above discussion demonstrates the key role of positive human wealth in generating a stationary distribution of wealth. According to Proposition 5, however, this is only one of two conditions that must be satisfied in order to ensure stationarity. The second of these

[18]In terms of equations (9) and (10) from the main model of Section 2, the term $\Lambda - \frac{1}{2}\Gamma^2$ is larger for poor households than for wealthy households in this case.

[19]This is a necessary consequence of adding the same value of $h(0)$ to different values of $w_i(s, T)$.

conditions requires that the strength of households' bequest motives χ not be too high. This condition ensures that each dynastic household's total accumulated wealth on average does not grow arbitrarily large. If households' wealth grows arbitrarily large, then asymptotically the growth rate of wealth for all households is unaffected by positive shocks to human wealth across generations, since fixed initial human wealth $h(0)$ is eventually dwarfed by growing total wealth $x_i(t)$. This outcome can also be seen from equation (50). According to this equation, if the bequest motive χ is large enough, then the process x_i will asymptotically behave like a continuous semimartingale with equal growth rates across households. In the two remarks in Appendix A, we show that such a process leads to an unstable distribution of wealth just like in equation (13) from Theorem 2.

An important implication of Proposition 5 is that the size of intergenerational transfers alone is essentially irrelevant to the stability of the distribution of wealth in models with idiosyncratic investment risk. Instead, stability depends on the size of intergenerational transfers as a proportion of dynastic households' wealth. In particular, stability requires that wealthy households transfer proportionally less total wealth (human wealth plus financial wealth) to their offspring than do poor households.[20] The absolute size of these transfers, however, is not directly relevant, since large transfers that are smaller proportions of total wealth for wealthy households will generate stability while small transfers that are larger proportions of total wealth for wealthy households will generate instability. In other words, it does not matter whether households leave 99% or 1% of their wealth to their offspring, only that wealthy households leave a smaller proportion than poor households.

This discussion elucidates the connection between our results and those of Benhabib et al. (2011), Benhabib and Zhu (2009), and Zhu (2010). In particular, the stability that is achieved in the setups of Benhabib et al. (2011) and Zhu (2010) through implicitly redistributive intergenerational transfers is a consequence of positive discounted labor incomes interacting with limited growth of household wealth due to a low intensity of bequest motives. Conversely, because the overlapping generations model of Benhabib and Zhu (2009) does not include labor income, intergenerational transfers in this setup are never redistributive and hence stability is achieved only in the presence of an explicit redistributive mechanism. The intensity of households' bequest motives and the size of their bequests to offspring are irrelevant in this case, just like in the previous paragraph's discussion.

[20] An implicit proof of this statement is provided by the two remarks in Appendix A. For a more direct proof in a setting that directly includes intergenerational transfers, see the appendix of Fernholz (2014).

Given the subtlety of the interaction between positive discounted labor incomes and the growth of household wealth in setups with overlapping generations, we believe that further research about the proportional value of intergenerational transfers for wealthy versus poor households is warranted. Indeed, when contrasted with Benhabib et al. (2011) and Zhu (2010), our results highlight the importance of different structural assumptions for the stability of the economy. In the real world, are intergenerational transfers an implicit redistributive mechanism? If so, empirically how important is this implicit redistributive mechanism relative to explicitly redistributive government tax and fiscal policies? The answer to these questions could have useful policy implications.

References

Angeletos, G.-M. (2007, January). Uninsured idiosyncratic investment risk and aggregate saving. *Review of Economic Dynamics 10*(1), 1–30.

Angeletos, G.-M. and L.-E. Calvet (2006, September). Idiosyncratic production risk, growth, and the business cycle. *Journal of Monetary Economics 53*(6), 1095–1115.

Atkinson, A. B. (2005, March). Top incomes in the UK over the 20th century. *Journal of the Royal Statistical Society: Series A (Statistics in Society) 168*(2), 325–343.

Atkinson, A. B., T. Piketty, and E. Saez (2011, March). Top incomes in the long run of history. *Journal of Economic Literature 49*(1), 3–71.

Becker, R. A. (1980, September). On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. *Quarterly Journal of Economics 95*(2), 375–382.

Benhabib, J. and A. Bisin (2007, September). The distribution of wealth: Intergenerational transmission and redistributive policies. *NYU Working Paper*.

Benhabib, J., A. Bisin, and S. Zhu (2011, January). The distribution of wealth and fiscal policy in economies with infinitely lived agents. *Econometrica 79*(1), 123–157.

Benhabib, J., A. Bisin, and S. Zhu (2014). The distribution of wealth in the blanchard-yaari model. *Macroeconomic Dynamics*, forthcoming.

Benhabib, J. and S. Zhu (2009, December). Age, luck, and inheritance. *NYU Working Paper*.

Bertaut, C. and M. Starr-McCluer (2002). Household portfolios in the United States. In L. Guiso, M. Haliassos, and T. Jappelli (Eds.), *Household Portfolios*. Cambridge, MA: MIT Press.

Cagetti, M. and M. De Nardi (2006, October). Entrepreneurship, frictions, and wealth. *Journal of Political Economy 114*(5), 835–870.

Cagetti, M. and M. De Nardi (2008, September). Wealth inequality: Data and models. *Macroeconomic Dynamics 12*(S2), 285–313.

Campbell, J. Y. and L. M. Viceira (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford, UK: Oxford University Press.

Carlyle, T. (1837). *The French Revolution: A History*. London, UK: Chapman & Hall.

Case, K. E. and R. J. Shiller (1989, March). The efficiency of the market for single-family homes. *American Economic Review 79*(1), 125–137.

Castañeda, A., J. Díaz-Giménez, and J.-V. Ríos-Rull (2003, August). Accounting for U.S. earnings and wealth inequality. *Journal of Political Economy 111*(4), 818–857.

Champernowne, D. G. (1953, June). A model of income distribution. *Economic Journal 63*(250), 318–351.

Davies, J. B., S. Sandström, A. Shorrocks, and E. N. Wolff (2011, March). The level and distribution of global household wealth. *Economic Journal 121*(551), 223–254.

De Nardi, M. (2004, July). Wealth inequality and intergenerational links. *Review of Economic Studies 71*(3), 743–768.

Diamond, P. and E. Saez (2011, Fall). The case for a progressive tax: From basic research to policy recommendations. *Journal of Economic Perspectives 25*(4), 165–190.

Díaz-Giménez, J., V. Quadrini, J.-V. Ríos-Rull, and S. B. Rodríguez (2002, Summer). Updated facts on the U.S. distributions of earnings, income, and wealth. *Federal Reserve Bank of Minneapolis Quarterly Review 26*(3), 2–35.

Fernholz, E. R. (2002). *Stochastic Portfolio Theory*. New York, NY: Springer-Verlag.

Fernholz, R. (2014, April). A model of economic mobility and the distribution of wealth. *Claremont McKenna College Working Paper*.

Fernholz, R. and I. Karatzas (2009). Stochastic portfolio theory: an overview. In A. Bensoussan and Q. Zhang (Eds.), *Mathematical Modelling and Numerical Methods in Finance: Special Volume, Handbook of Numerical Analysis*, Volume XV, pp. 89–168. Amsterdam: North-Holland.

Flavin, M. and T. Yamashita (2002, March). Owner-occupied housing and the composition of the household portfolio. *American Economic Review 92*(1), 345–362.

Gabaix, X. (2009, 05). Power laws in economics and finance. *Annual Review of Economics 1*(1), 255–294.

Goldie, C. M. (1991, February). Implicit renewal theory and tails of solutions of random equations. *Annals of Applied Probability 1*(1), 126–166.

Hendricks, L. (2007, September). How important is discount rate heterogeneity for wealth inequality? *Journal of Economic Dynamics and Control 31*(9), 3042–3068.

Ichiba, T., V. Papathanakos, A. Banner, I. Karatzas, and R. Fernholz (2011). Hybrid atlas models. *Annals of Applied Probability 21*(2), 609–644.

Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus*. New York, NY: Springer-Verlag.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Mathematica 131*(1), 207–248.

Krussel, P. and A. A. Smith (1998, October). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867–896.

Kumhof, M., R. Rancière, and P. Winant (2013, June). Inequality, leverage, and crises. *IMF Working Paper*.

Ljungqvist, L. and T. J. Sargent (2004). *Recursive Macroeconomic Theory*. Cambridge, MA: MIT Press.

Merton, R. C. (1969, August). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics 51*(3), 247–257.

Moriguchi, C. and E. Saez (2008, November). The evolution of income concentration in Japan, 1886-2005: Evidence from income tax statistics. *Review of Economics and Statistics 90*(4), 713–734.

Moskowitz, T. J. and A. Vissing-Jorgensen (2002, September). The returns to entrepreneurial investment: A private equity premium puzzle? *American Economic Review 92*(4), 745–778.

Piketty, T. (2003, October). Income inequality in France, 1901-1998. *Journal of Political Economy 111*(5), 1004–1042.

Piketty, T. and E. Saez (2003, Feburary). Income inequality in the United States, 1913-1998. *Quarterly Journal of Economics 118*(1), 1–39.

Quadrini, V. (2000, January). Entrepreneurship, saving, and social mobility. *Review of Economic Dynamics 3*(1), 1–40.

Rajan, R. (Ed.) (2010). *Fault Lines: How Hidden Fractures Still Threaten the World Economy.* Princeton, NJ: Princeton University Press.

Saez, E. and M. R. Veall (2005, June). The evolution of high incomes in Northern America: Lessons from Canadian evidence. *American Economic Review 95*(3), 831–849.

Stiglitz, J. E. (1969, August). Distribution of income and wealth among individuals. *Econometrica 37*(3), 382–397.

Wolff, E. N. (2010, March). Recent trends in household wealth in the united states: Rising debt and the middle-class squeeze—an update to 2007. *Levy Economics Institute of Bard College Working Paper No. 589.*

Wolff, E. N. (2012). Changes in household wealth in the 1980s and 1990s in the U.S. In E. N. Wolff (Ed.), *International Perspectives on Household Wealth.* Elgar Publishing Ltd., forthcoming.

Zhu, S. (2010, November). Wealth distribution under idiosyncratic investment risk. *NYU Working Paper.*

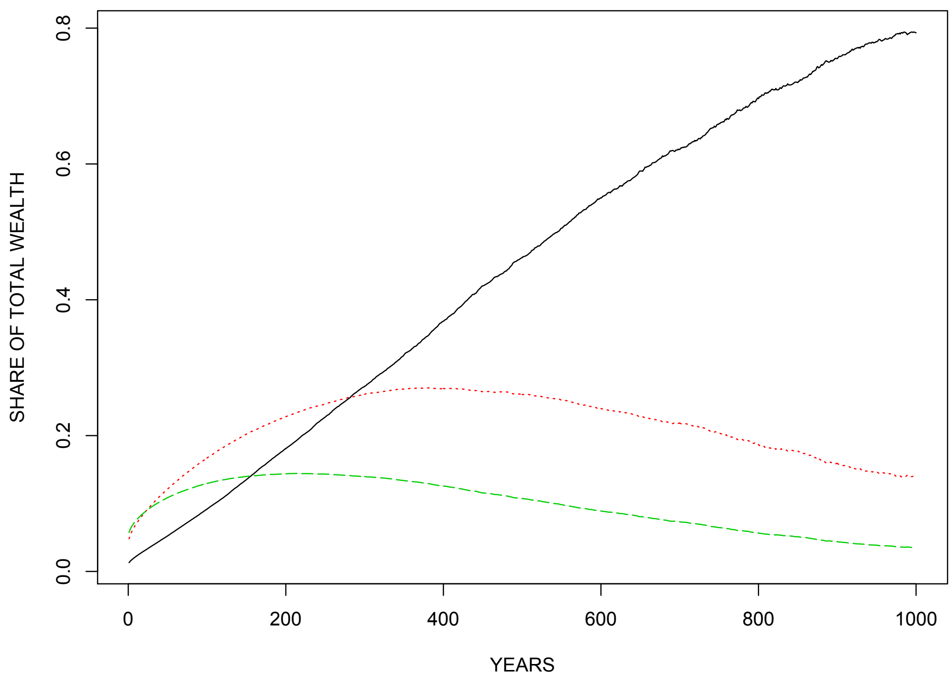

Figure 1: The shares of total wealth held by the wealthiest 1% (solid black line), the wealthiest 1-5% (dotted red line), and the wealthiest 5-10% (dashed green line). ($\Gamma = 0.1$)

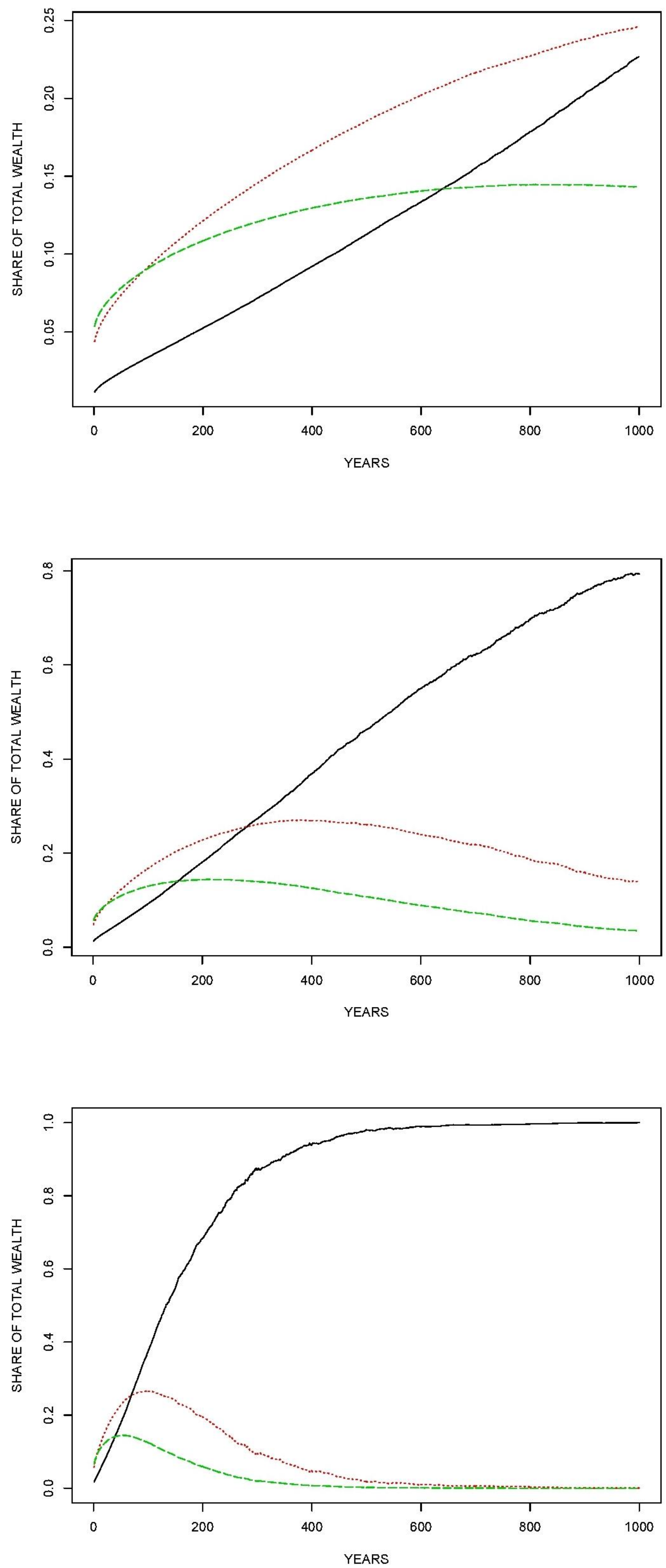

Figure 2: The shares of total wealth held by the wealthiest 1% (solid black line), the wealthiest 1-5% (dotted red line), and the wealthiest 5-10% (dashed green line) for different values of Γ. (Top: $\Gamma = 0.05$, Middle: $\Gamma = 0.1$, Bottom: $\Gamma = 0.2$)

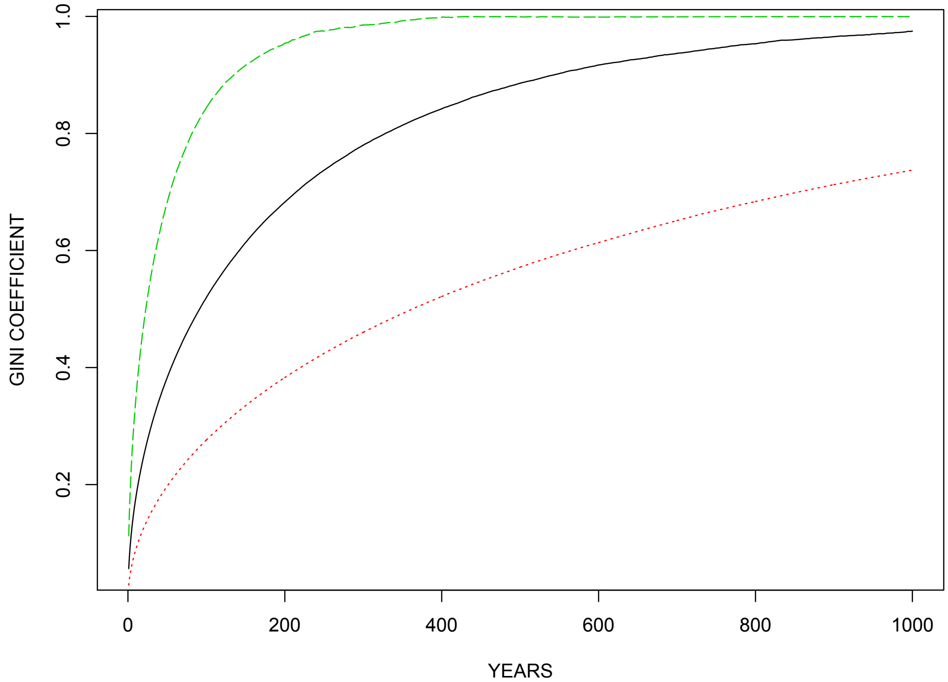

Figure 3: The Gini coefficient of the economy for $\Gamma = 0.1$ (solid black line), $\Gamma = 0.05$ (dotted red line), and $\Gamma = 0.2$ (dashed green line).

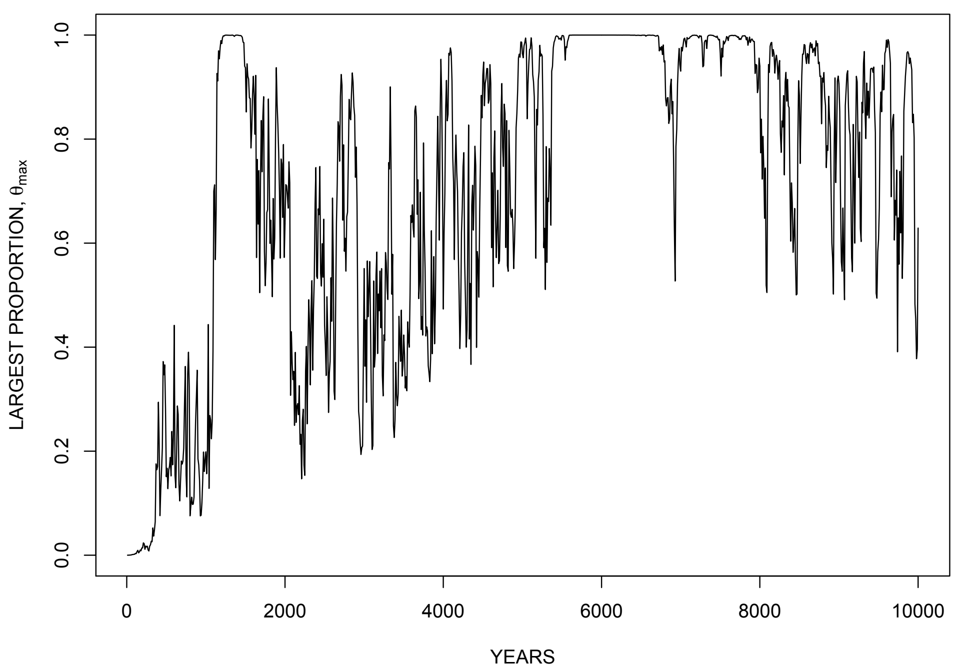

Figure 4: The share of total wealth held by the wealthiest single household. ($\Gamma = 0.2$)