Econ 207 Lecture Notes

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1 Subjective Expected Utility: Anscombe-Aumann

Lecture 2, January 24: The goal of this section is to model subjective uncertainty (i.e. "horse race" lotteries where the probability of each outcome can vary across individuals) and present the main Anscombe-Aumann result about such uncertainty. We start with a finite set X of outcomes and a corresponding set $\Delta(X)$ of lotteries on X. There is also a finite set of states of nature $S = \{1, 2, \ldots, S\}$. An act is defined as a function $f : S \to \Delta(X)$ specifying for each $s \in S$ a distribution $f_s = f(s) \in \Delta(X)$ obtaining in state $s \in S$. It follows then that $f_s(x)$ is the probability of outcome $x \in X$ occurring conditional on state $s \in S$ being realized. Next, we define $H = \Delta(X)^S$ as the set of all possible acts and the subset H_c of H by

$$H_c = \{f \in H : f(s) = f(s') \quad \forall s, s' \in S\}.$$

This is the set of constant acts; regardless of the state of nature, the same lottery (distribution) obtains. Note that the set of constant acts H_c can be associated with $\Delta(X)$ since there is some constant act that maps to any element of $\Delta(X)$. Finally, in this model there is of course a decision-maker whose primitive is a preference relation over the set of acts H. This preference relation $\succsim$ is a binary relation on H that is complete and transitive (a weak order).

We are now ready to prove some results, starting with the classic mixture space theorem. First we need some definitions.

Definition 1.1. A mixture space is a set Π and a family of functions $m_a : \Pi \times \Pi \to \Pi$, $a \in [0, 1]$ such that

$$m_1(\pi, \rho) = \pi \qquad \forall (\pi, \rho) \in \Pi \times \Pi$$

$$m_a(\pi, \rho) = m_{1-a}(\rho, \pi) \qquad \forall (\pi, \rho) \in \Pi \times \Pi \quad \text{and} \quad \forall a \in [0, 1]$$

$$m_a(m_b(\pi, \rho)) = m_{ab}(\pi, \rho) \qquad \forall (\pi, \rho) \in \Pi \times \Pi \quad \text{and} \quad \forall a, b \in [0, 1].$$

Some examples of mixture spaces include some convex subset $\Pi \subset \mathbb{R}^k$ with $m_a(\pi, \rho) = a\pi + (1 - a)\rho$ for any $a \in [0, 1]$, $\Pi = \Delta(X) \subset \mathbb{R}^n$ where m_a is simply convex combinations of lotteries, and finally $\Pi = H = \Delta(X)^S$ which is a convex subset of $(\mathbb{R}^n)^S$ where $m_a(f,g) = af + (1-a)g \equiv af(s) + (1-a)g(s)$ for any $a \in [0,1]$.

Definition 1.2. Let $(\Pi; m_a, a \in [0,1])$ be a mixture space and $\succsim$ be a preference relation on Π. Then, $\succsim$ is independent if for all $\pi, \rho, \sigma \in \Pi$ and all $a \in (0,1)$, $\pi \succsim \rho$ if and only if $m_a(\pi,\sigma) \succsim m_a(\rho,\sigma)$. Additionally, $\succsim$ is Archimedean if for all $\pi, \rho, \sigma \in \Pi$ such that $\pi \succsim \rho \succsim \sigma$, there exist $a, b \in (0,1)$ such that $m_a(\pi,\sigma) \succ \rho \succ m_b(\pi,\sigma)$.

Theorem 1.3. *(Mixture Space Theorem) Let $(\Pi; m_a, a \in [0,1])$ be a mixture space and $\succsim$ be a preference relation on Π. Then, $\succsim$ is independent and Archimedean if and only if there exists a function $F : \Pi \to \mathbb{R}$ representing $\succsim$ (so $\pi \succsim \rho$ iff $F(\pi) \geq F(\rho)$ for all $\pi, \rho \in \Pi$) such that for all $\pi, \rho \in \Pi$ and $a \in [0,1]$, $F(m_a(\pi,\rho)) = aF(\pi) + (1-a)F(\rho)$. Moreover, F is unique up to positive affine transformations.*

The proof of the mixture space theorem is rather tedious and not particularly instructive or interesting. However, it is essential for the main result in this section, the Anscombe-Aumann expected utility theorem. Before stating and proving this theorem, we need one more basic definition.

Definition 1.4. A preference relation $\succsim$ on H is monotonic if for all $f, g \in H$, $f_s \succsim g_s$ for all $s \in S$ implies that $f \succsim g$. It is nondegenerate if there exist $f, g \in H$ such that $f \succ g$.

Note that the statement $f_s \succsim g_s$ refers to preferences over lotteries (elements of $\Delta(X)$). This makes sense because the set of constant acts H_c, together with standard preference relation over acts, induces a preference relation over lotteries. In other words, for $x, y \in \Delta(X)$, $x \succsim y$ if and only if the constant act that maps all states to x is weakly preferred to the constant act that maps all states to y. We see then that monotonicity simply says that if the lotteries in every state for some act are preferred to those for some other act, then that first act must be preferred to the second.

Theorem 1.5. *(Anscombe-Aumann Expected Utility Theorem) A preference relation $\succsim$ on H is independent, Archimedean, monotonic, and nondegenerate if and only if there exists a von Neumann-Morgenstern index $u : X \to \mathbb{R}$ and a unique subjective probability measure $\mu \in \Delta(S)$ such that*

$$U(f) = \sum_{s \in S} \mu_s \left(\sum_{x \in X} f_s(x) u(x) \right)$$

represents $\succsim$. Moreover, u is unique up to positive affine transformations.

We only sketch out a proof of the theorem here. Denote by $\succsim_{\Delta(X)}$ the binary relation on $\Delta(X)$ induced by $\succsim$. Thus, for $p, q \in \Delta(X)$, $p \succsim_{\Delta(X)} q$ if and only if $(p, p, \ldots, p) \succsim (q, q, \ldots, q)$. $\succsim_{\Delta(X)}$ is a preference relation on $\Delta(X)$ that is independent and Archimedean, so by the mixture space theorem 1.3 we know there exists a linear function $v : \Delta(X) \to \mathbb{R}$ representing $\succsim_{\Delta(X)}$.

Then, $v(\Delta(X))$ is a nonempty interval in $\mathbb{R}$. Without loss of generality, take $0 \in \text{Int}(v(\Delta(X)))$. Next, construct an order $\geq^*$ on the set $v(\Delta(X))^S$ (note that this space is a mixture space). Define $\geq^*$ on $v(\Delta(X))^S$ for $w, y \in v(\Delta(X))^S \subset \mathbb{R}^S$ by $w \geq^* y$ if and only if there exist $f, g \in H$ such that $w = v \circ f$, $y = v \circ g$, and $f \succsim g$. Note that $v \circ f = (v(f_1), v(f_2), \ldots, v(f_S))$. Then $\geq^*$ is well-defined. To see this, let $f, f', g, g' \in H$ such that $w = v \circ f = v \circ f'$ and $y = v \circ g = v \circ g'$. Then $v(f_s) = v(f'_s)$ and $v(g_s) = v(g'_s)$ for all $s \in S$. But that implies that $f_s \sim_{\Delta(X)} f'_s$ and $g_s \sim_{\Delta(X)} g'_s$ for all $s \in S$, which by monotonicity implies that $f \sim f'$ and $g \sim g'$.

It is straightforward to show that $\geq^*$ is a preference relation on $v(\Delta(X))^S$ that is independent and Archimedean. Furthermore, $\geq^*$ is monotonic in the sense that if $w, y \in v(\Delta(X))^S$ and $w \geq y$ (in the ordinary, pointwise sense in $\mathbb{R}^S$) then $w \geq^* y$. Thus, by the mixture space theorem there exists a linear function representing $\geq^*$ on $v(\Delta(X))^S$. By monotonicity, all coefficients of this function are nonnegative and hence there exists some $c \in \mathbb{R}^S_+$, with $c \neq 0$ (by nondegeneracy), such that $w \geq^* y$ if and only if $c \cdot w \geq c \cdot y$. Of course, this is equivalent to

$$\left(\frac{1}{\sum_{s \in S} c_s}\right) c \cdot w \geq \left(\frac{1}{\sum_{s \in S} c_s}\right) c \cdot y,$$

which gives us the desired subjective distribution over S:

$$\mu = \left(\frac{1}{\sum_{s \in S} c_s}\right) \in \Delta(S).$$

2 Gilboa-Schmeidler Maxmin Expected Utility

Lecture 3, January 31: The example that most inspired extensions and departures from the subjective expected utility framework is the Ellsberg paradox. It states that when decision makers are to draw one ball at random from an urn containing 30 red balls and 60 balls that are either blue or yellow, they generally prefer to bet on a red ball being drawn rather than a blue (or yellow) ball being drawn. In addition, decision makers prefer to bet on a blue or yellow ball being drawn (both at the same time now) rather than a red or blue ball being drawn (same for red and yellow of course).

Preferences of this kind, which have been confirmed by experiments, violate the axiom of independence that is crucial to the Anscombe-Aumann theory described above. They also rule out the existence of a consistent probability distribution like the one established in the proof of the expected utility theorem. More precisely, given the state space $S = \{r, b, y\}$ and the preference relation $\succsim$ over lotteries in this space, we can define a relation $\dot{\succsim}$ over events in the state space by $E \,\dot{\succsim}\, E'$ if and only if $h^E \succsim h^{E'}$, where h^E is the lottery that pays \$100 if the event $E \subset S$ occurs and \$0 otherwise. Ellsberg preferences imply that $\{r\} \,\dot{\succ}\, \{b\}$ and $\{b, y\} \,\dot{\succ}\, \{r, y\}$ in this case, and this requires that $\pi(r) > \pi(b)$ and $\pi(b, y) > \pi(r, y)$ for any probability distribution $\pi \in \Delta(S)$ that is to be consistent. But because $\pi(r) = 1 - \pi(b, y)$ and $\pi(b) = 1 - \pi(r, y)$, it follows that no such distribution is possible.

It is precisely this issue that Gilboa and Schmeidler (1989) address. They adapt the Anscombe-Aumann setup where X is a finite set of outcomes and $S = \{1, 2, \ldots, S\}$ is the finite state space, but now the standard independence axiom is replaced by the weaker condition of certainty independence.

Definition 2.1. A preference relation $\succsim$ on H is certainty independent if for all $f, g \in H$, all $p \in H_c$, and all $\alpha \in (0, 1)$, $f \succsim g$ if and only if $\alpha f + (1 - \alpha)p \succsim \alpha g + (1 - \alpha)p$. This preference relation is uncertainty averse if $f \sim g$ if and only if $\alpha f + (1 - \alpha)g \succsim f \sim g$.

Theorem 2.2. *(Gilboa-Schmeidler Maxmin Expected Utility Theorem) A preference relation $\succsim$ on H is nondegenerate, monotonic, Archimedean, certainty independent, and uncertainty averse if and only if there exists a von Neumann-Morgenstern index $u : X \to \mathbb{R}$ and a unique closed and convex set $M \subset \Delta(S)$, such that*

$$U(f) = \min_{\mu \in M} \sum_{s \in S} \mu_s \left(\sum_{x \in X} f_s(x) u(x) \right)$$

represents $\succsim$. Moreover, u is unique up to positive affine transformations.

Before proving this theorem, some basic results from convex analysis are necessary.

Definition 2.3. A set $K \subset \mathbb{R}^n$ is a convex cone if (i) for all $x \in K$ and all $\lambda > 0$, $\lambda x \in K$, and (ii) for all $x, y \in K$ and all $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in K$.

Definition 2.4. Let K be a nonempty convex cone. The dual cone of K, denoted by K^0, is the collection of all normal vectors to K at 0: $K^0 = \{y \in \mathbb{R}^n : y \cdot x \geq 0 \quad \forall x \in K\}$.

A basic result of convex analysis says that if $K \subset \mathbb{R}^n$ is a nonempty, closed, convex cone, then K^0 is also a nonempty, closed, convex cone and $K^{00} = K$. An implication of this result is that $x \in K$ if and only if $y \cdot x \geq 0$ for all $y \in K^0$. We are now ready to prove the Gilboa-Schmeidler theorem. Throughout the proof, for $a \in \mathbb{R}$, let $\bar{a} = (a, a, \ldots, a) \in \mathbb{R}^s$.

As in the proof of the Anscombe-Aumann theorem, define the induced preference relation $\succsim_{\Delta(X)}$ on $\Delta(X)$ by restricting $\succsim$ to H_c. Then $\succsim_{\Delta(X)}$ is Archimedean and independent (by independence of $\succsim$). By the mixture space theorem 1.3, there exists a linear function $v : \Delta(X) \to \mathbb{R}$, unique up to positive affine transformations, representing $\succsim_{\Delta(X)}$. As before, $v(\Delta(X))$ is an interval in $\mathbb{R}$ and we can take $0 \in \text{Int}(v(\Delta(X)))$ without loss of generality.

Then we define the binary relation $\geq^*$ on $v(\Delta(X))^S$ by for $w, y \in v(\Delta(X))^S$, $w \geq^* y$ if and only if there exist $f, g \in H$ such that $w = v \circ f$, $y = v \circ g$, and $f \succsim g$. This is a well-defined preference relation on $v(\Delta(X))^S$ that is Archimedean and independent. Furthermore, by certainty independence, for all $w, y \in v(\Delta(X))^S$, all $a \in v(\Delta(X))$, and all $\alpha \in (0,1)$, $w \geq^* y$ if and only if $\alpha w + (1-\alpha)\bar{a} \geq^* \alpha y + (1-\alpha)\bar{a}$. By uncertainty aversion, $w =^* y$ if and only if $\alpha w + (1-\alpha)y \geq^* w =^* y$. We can extend $\geq^*$ to $\mathbb{R}^S$ by noting that for all $w, y \in v(\Delta(X))^S$ and all $\lambda \in (0,1]$, $w \geq^* y$ if and only if $\lambda w \geq^* \lambda y$ (just mix with $\bar{0}$ and use certainty independence). In particular, given $w, y \in \mathbb{R}^S$, let $w \geq^* y$ if and only if there exists some $\lambda \in (0,1]$ such that $\lambda w, \lambda y \in v(\Delta(X))^S$ and $\lambda w \geq^* \lambda y$.

It follows now that $\geq^*$ is an Archimedean preference relation on $\mathbb{R}^S$ such that whenever $w \geq y$, $w \geq^* y$ as well (by monotonicity). It can be shown (which we did for homework I think) that for any $w \in \mathbb{R}^S$, there exists a unique $a_w \in \mathbb{R}$ such that $w =^* \overline{a_w}$. We call this the certainty equivalent of w. It can also be shown (also done in homework) that for all $w \in \mathbb{R}^S$ and all $a, b \in \mathbb{R}$, $w \geq^* \bar{a}$ if and only if $w - \bar{b} \geq^* \bar{a} - \bar{b}$. From this it follows that

$$\{z \in \mathbb{R}^S : z \geq^* w\} = \{z \in \mathbb{R}^S : z - \overline{a_w} \geq^* \bar{0}\} = \{z \in \mathbb{R}^S : z \geq^* \bar{0}\} + \{\overline{a_w}\}.$$

In other words, all $\geq^*$-better-than sets are translates of $\{z \in \mathbb{R}^S : z \geq^* \bar{0}\}$ along some diagonal.

We claim that this set, $\{z \in \mathbb{R}^S : z \geq^* \bar{0}\}$, is a nonempty, closed, convex cone. First, note that by monotonicity of $\geq^*$ this set is nonempty. Next, let $w, y \geq^* \bar{0}$ and $\alpha \in (0,1)$ and choose $a_w, a_y \in \mathbb{R}$ such that $w - \overline{a_w} =^* y - \overline{a_y} =^* \bar{0}$ (note that by monotonicity $a_w, a_y \geq 0$). It follows by uncertainty aversion that $\alpha w + (1-\alpha)y \geq^* \alpha w + (1-\alpha)y - [\alpha\overline{a_w} + (1-\alpha)\overline{a_y}] = \alpha(w - \overline{a_w}) + (1-\alpha)(y - \overline{a_y}) \geq^* \bar{0}$, and so we have shown convexity. This set is a cone by construction since $y \geq^* \bar{0}$ if and only if $\lambda y \geq^* \lambda 0 = 0$ for all $\lambda > 0$. The set is also closed, completing the proof of our claim. Note that by monotonicity we have $\mathbb{R}^S_+ \subset \{z \in \mathbb{R}^S : z \geq^* \bar{0}\}$.

Lecture 4, February 7: Let $D = \{z \in \mathbb{R}^S : z \geq^* \bar{0}\}^0 = \{d \in \mathbb{R}^S : d \cdot z \geq 0 \quad \forall z \geq^* \bar{0}\}$. It follows that $D \subset \mathbb{R}^S_+$. Now, set $M = D \cap \Delta(S)$. Note that for any $z \in \mathbb{R}^S$, $z \geq^* \bar{0}$ if and only if $d \cdot z \geq 0$ for all $d \in D$, which is true if and only if $\frac{1}{\sum_s d_s}(d \cdot z) \geq 0$ for all $d \in D \backslash \{0\}$, which is equivalent to $\mu \cdot z \geq 0$ for all $\mu \in M$. Given this, it follows that for any $w, y \in \mathbb{R}^S$, $w \geq^* y$ if and only if $\mu \cdot (w - \overline{a_y}) \geq 0$ for all $\mu \in M$. This is equivalent to $\mu \cdot w \geq a_y$ for all $\mu \in M$, which is equivalent to $\min_{\mu \in M}(\mu \cdot w) \geq a_y$.

From this, it is clear that if we can show that $a_y = \min_{\mu \in M} \mu \cdot y$, then we are done. Note that $y - \overline{a_y} =^* \bar{0}$ and for all $b > a_y$, $y - \bar{b} <^* \bar{0}$. This implies that for all $b > a_y$, there exists some $\mu \in M$ such that $\mu \cdot (y - \bar{b}) < 0$, or equivalently $\mu \cdot y < b$. But this means that $\min_{\mu \in M} \mu \cdot y \leq a_y$, which of course implies that $\min_{\mu \in M} \mu \cdot y = a_y$.

Finally, then, we have shown that for any $w, y \in \mathbb{R}^S$, $w \geq^* y$ if and only if $\min_{\mu \in M} \mu \cdot w \geq \min_{\mu \in M} \mu \cdot y$. Therefore, for any $f, g \in H$, we have that $f \succsim g$ if and only if $\min_{\mu \in M} \mu \cdot (v \circ f) \geq \min_{\mu \in M} \mu \cdot (v \circ g)$, which is obviously equivalent to the statement

$$\min_{\mu \in M} \sum_{s \in S} \mu_s \left(\sum_{x \in X} f_s(x) u(x) \right) \geq \min_{\mu \in M} \sum_{s \in S} \mu_s \left(\sum_{x \in X} g_s(x) u(x) \right),$$

where $u : X \to \mathbb{R}$ is a unique (up to positive affine transformations) von Neumann-Morgenstern index given by $u(x) = v(k_x)$, where $k_x \in \Delta(X)$ is the lottery that yields state $x \in X$ with certainty.

3 Choquet Expected Utility

Another important extension of the Anscombe-Aumann subjective expected utility theory that does not conflict with the behavior described by the Ellsberg paradox is the capacity-based approach developed by Schmeidler (1989). Before proving the main result, we need several definitions and examples. As always, we assume a finite set of outcomes X and a finite set of states $S = \{1, 2, \dots, S\}$.

Definition 3.1. Acts $f, g \in H$ are comonotonic for $\succsim$ if there are no states $s, s' \in S$ such that $f_s \succ f_{s'}$ and $g_{s'} \succ g_s$. A preference relation $\succsim$ on H is comonotonic independent if for all $f, g, h \in H$ that are pairwise comonotonic and for all $\alpha \in (0, 1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

Note that $f_s \succ f_{s'}$ and $g_{s'} \succ g_s$ refer to preferences over lotteries as defined and explained in section one above. Clearly, all constant acts are comonotonic with any act $h \in H$. Also, the preferences described in the Ellsberg paradox are consistent with comonotonic independence.

Definition 3.2. A capacity on S is a function $\nu : 2^S \to [0, 1]$ such that $v(\varnothing) = 0$, $v(S) = 1$, and for any $A, B \subset S$ such that $A \subset B$, $\nu(A) \leq \nu(B)$. If in addition, $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$ for all $A, B \subset S$, then ν is a probability measure.

For example, if $S = \{1, 2\}$ then a capacity could be given by $\nu(\varnothing) = 0$, $\nu(S) = 1$, $\nu(1) = 1/3$, and $\nu(2) = 1/3$, or instead by $\nu(1) = 2/3$ and $\nu(2) = 4/5$. Alternatively, if $S = \{1, 2, 3\}$ then $\nu(\varnothing) = 0$, $\nu(S) = 1$, $\nu(1) = \nu(2) = \nu(3) = 1/4$, and $\nu(1, 2) = \nu(2, 3) = \nu(1, 3) = 2/3$ defines a capacity.

We move now to the topic of Choquet expectations. Let $w \in \mathbb{R}^S$ with $w_1 < w_2 < \cdots < w_S$. If $\mu \in \Delta(S)$, then

$$\begin{aligned} E_\mu(w) = \sum_s \mu_s w_s &= w_1 + (w_2 - w_1) \sum_{s=2}^{S} \mu_s + (w_3 - w_2) \sum_{s=3}^{S} \mu_s + \cdots + (w_S - w_{S-1}) \mu_S \\ &= w_1 + \sum_{s=2}^{S} \left[(w_s - w_{s-1}) \sum_{t=s}^{S} \mu_t \right] = w_1 + \sum_{s=2}^{S} (w_s - w_{s-1}) \mu(s, \dots, S). \end{aligned}$$

Alternatively, we can write

$$E_\mu(w) = \sum_s \mu_s w_s = \sum_{s=1}^{S-1} [\mu(s, \dots, S) - \mu(s+1, \dots, S)] w_s + \mu_S w_S.$$

The definition of the Choquet expectation of w is related to these two expressions. In particular, for a given capacity ν on S, we have

$$\mathrm{CE}_\nu(w) = w_1 + \sum_{s=2}^{S} (w_s - w_{s-1}) \nu(s, \dots, S) = \sum_{s=1}^{S-1} w_s [\nu(s, \dots, S) - \nu(s+1, \dots, S)] + w_S \nu(S).$$

In the general case where it is not necessarily the case that $w_1 < w_2 < \cdots < w_S$, things are a little more complicated. Let $\mathcal{P}(S)$ be the set of all permutations of S (so bijections from S to S) and given some $\rho \in \mathcal{P}(S)$ define $A_t^\rho = \{\rho(t), \rho(t+1), \ldots, \rho(S)\}$ for each $t \in S$. Also, let $R(w) = \{\pi \in \mathcal{P}(S) : w_{\rho(1)} \leq w_{\rho(2)} \leq \cdots \leq w_{\rho(S)}\}$. Then, for any $\rho \in R(w)$ we have

$$\mathrm{CE}_\nu(w) = \sum_{s=1}^{S-1} w_{\rho(s)} \left[\nu(A_s^\rho) - \nu(A_{s+1}^\rho)\right] + w_{\rho(S)}\nu(\rho(S)).$$

An example is in order. Let $S = \{1, 2\}$, $\nu : 2^S \to [0, 1]$ be a capacity on S, and $\mathbb{R}_+ = X \subset \mathbb{R}$ be the set of prizes. Also, let $u : X \to \mathbb{R}$ by a von Neumann-Morgenstern index that is strictly increasing and concave. If $x = (x_1, x_2)$ and $x_1 < x_2$, then $u(x_1) < u(x_2)$ and

$$\mathrm{CE}_\nu(u \circ x) = u(x_1) + [u(x_2) - u(x_1)]\nu_2 = (1 - \nu_2)u(x_1) + \nu_2 u(x_2), \tag{1}$$

where $\nu_1 = \nu(1)$ and $\nu_2 = \nu(2)$. Alternatively, if $x_1 \geq x_2$, then $u(x_1) \geq u(x_2)$ and

$$\mathrm{CE}_\nu(u \circ x) = u(x_2) + [u(x_1) - u(x_2)]\nu_1 = \nu_1 u(x_1) + (1 - \nu_1)u(x_2). \tag{2}$$

Definition 3.3. A capacity ν on S is convex if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all $A, B \subset S$. In particular, if ν is convex, then for all A, B such that $A \cap B = \varnothing$, $\nu(A \cup B) \geq \nu(A) + \nu(B)$.

Definition 3.4. The core of a capacity ν on S is given by

$$\mathrm{Core}(\nu) = \{\mu \in \Delta(S) : \mu(A) \geq \nu(A) \quad \forall A \subset S\}.$$

A couple of things are worth noting here. First, if ν is a convex capacity, then there exists some $\mu \in \Delta(S)$ such that $\mu(A) \geq \nu(A)$ for all $A \subset S$. Second, the core of a capacity ν is a compact, convex subset of $\Delta(S)$ (it can be empty in some cases). For example, letting $S = \{1, 2\}$, $\nu(\varnothing) = 0$, $\nu(S) = 1$, $\nu(1) = 1/3$, and $\nu(2) = 1/4$, we find that

$$\mathrm{Core}(\nu) = \{\mu \in \Delta(S) : \mu(1) \geq 1/3, \mu(2) \geq 1/4\} = \{\mu \in \Delta(S) : \mu(1) \in [1/3, 3/4]\}.$$

To see that the core can be empty, consider $S = \{1, 2\}$, $\tilde{\nu}(\varnothing) = 0$, $\tilde{\nu}(S) = 1$, $\tilde{\nu}(1) = 1/2$, and $\tilde{\nu}(2) = 3/4$. In this case, we have

$$\mathrm{Core}(\tilde{\nu}) = \{\mu \in \Delta(S) : \mu(1) \geq 1/2, \mu(2) \geq 3/4\} = \varnothing.$$

Lecture 5, February 14: Recalling the formula for the Choquet expectation (1), we find that if ν is as in the example just given above and $w \in \mathbb{R}^2$ is such that $w_1 \leq w_2$ (this is without loss of generality), then

$$\text{CE}_\nu(w) = w_1 + (w_2 - w_1)\nu_2 = w_1 + (w_2 - w_1)(1/4) = (3/4)w_1 + (1/4)w_2.$$

Of course, this expression is equivalent to $E_\mu(w)$, where $\mu \in \text{Core}(\nu)$ is such that $\nu(1) = 3/4$ and $\nu(2) = 1/4$. Furthermore, $E_\mu(w) = \min_{\mu \in \text{Core}(\nu)} E_\mu(w)$, an observation that leads us to our next result.

Theorem 3.5. *Let S be finite and ν be a convex capacity on S. Then, $\text{Core}(\nu)$ is a nonempty, convex, compact, subset of $\Delta(S)$ and for each $w \in \mathbb{R}^S$, $\text{CE}_\nu(w) = \min_{\mu \in \text{Core}(\nu)} E_\mu(w)$.*

Let ν be a convex capacity on S. Without loss of generality, fix $w \in \mathbb{R}^S$ such that $w_1 \leq w_2 \leq \cdots \leq w_S$. Then, letting $w_0 = 0$ we have $\text{CE}_\nu(w) = \sum_s (w_s - w_{s-1})\nu(s, \ldots, S)$. Define $E_s = \{s, s+1, \ldots, S\}$ with $E_0 = \varnothing$, and $d_s = w_s - w_{s-1}$ with $d_1 = w_1$. Then, $w = \sum_s d_s 1_{E_s}$ and $\text{CE}_\nu(w) = \sum_s d_s \nu(E_s)$.

Now, define $\mu \in \Delta(S)$ by $\mu(s) = \nu(E_s) - \nu(E_{s+1})$ for all $s \in S$. It is easy to see that $\mu(\varnothing) = 0$, $\mu(S) = 1$, and $\mu(s) \geq 0$ for all $s \in S$ (because ν is nondecreasing). Furthermore, it can easily be shown that $\mu(E_s) = \nu(E_s)$ for any $s \in S$, and we already showed in homework that $\mu \in \text{Core}(\nu)$. It follows that for any $\tilde{\mu} \in \text{Core}(\nu)$, we have

$$E_{\tilde{\mu}}(w) = \sum_{s=1}^{S} d_s \tilde{\mu}(E_s) \geq \sum_{s=1}^{S} d_s \nu(E_s) = \sum_{s=1}^{S} d_s \mu(E_s) = E_\mu(w) = \text{CE}_\nu(w),$$

which completes the proof that $\text{CE}_\nu(w) = \min_{\mu \in \text{Core}(\nu)} E_\mu(w)$.

Before moving on to the main theorem of this section, a few more interesting examples of capacities are worth mentioning. The first is called ϵ-contamination. Given an $\epsilon \in [0, 1]$ and $\mu \in \Delta(S)$, the capacity ν_ϵ is defined so that for any $A \subset S$,

$$\nu_\epsilon(A) = \begin{cases} 1 & \text{if } A = S, \\ (1 - \epsilon)\mu(A) & \text{if } A \neq S. \end{cases}$$

For all $\epsilon \in [0, 1]$, ν_ϵ is a convex capacity and its core is given by

$$\text{Core}(\nu_\epsilon) = \{\tilde{\mu} \in \Delta(S) : \tilde{\mu} = (1 - \epsilon)\mu + \epsilon\mu' \quad \text{for some } \mu' \in \Delta(S)\}.$$

The second example is called belief distortion. Given some $\mu \in \Delta(S)$, define ν by $\nu(A) = \varphi(\mu(A))$ for all $A \subset S$, where $\varphi : [0,1] \to [0,1]$ with $\varphi(0) = 0$, $\varphi(1) = 1$, and φ nondecreasing.

Theorem 3.6. *(Choquet Expected Utility Theorem) A preference relation $\succsim$ on H is Archimedean, monotonic, nondegenerate, and comonotonic independent if and only if there exists a von Neumann-Morgenstern index $u : X \to \mathbb{R}$, unique up to positive affine transformations, and a unique capacity ν on S, such that for all $f, g \in H$, $f \succsim g$ if and only if $\mathrm{CE}_\nu(\bar{u} \circ f) \geq \mathrm{CE}_\nu(\bar{u} \circ g)$ where, for $h \in H$,*

$$\bar{u} \circ h = \left(\sum_x h_1(x)u(x), \sum_x h_2(x)u(x), \ldots, \sum_x h_S(x)u(x) \right).$$

If in addition $\succsim$ is uncertainty averse, then ν is a convex capacity and for all $f, g \in H$, $f \succsim g$ if and only if

$$\min_{\mu \in \mathrm{Core}(\nu)} E_\mu(\bar{u} \circ f) \geq \min_{\mu \in \mathrm{Core}(\nu)} E_\mu(\bar{u} \circ g).$$

As in the proofs of theorems 1.5 and 2.2, define the preference relation $\succsim_{\Delta(X)}$ on $\Delta(X)$ by restricting $\succsim$ to H_c. By comonotonic independence (and because constant acts are comonotonic), $\succsim_{\Delta(X)}$ is Archimedean and independent and hence by the mixture space theorem 1.3 there exists a unique (up to positive affine transformations) linear function $v : \Delta(X) \to \mathbb{R}$ representing $\succsim_{\Delta(X)}$. Also as before, construct the Archimedean preference relation $\geq^*$ on $v(\Delta(X))^S$. Recall that $v(\Delta(X))$ is an interval in $\mathbb{R}$, and without loss of generality $0 \in \mathrm{Int}(v(\Delta(X)))$.

For each $\rho \in \mathcal{P}(S)$, set $C(\rho) = \{w \in \mathbb{R}^S : w_{\rho(1)} \leq w_{\rho(2)} \leq \cdots \leq w_{\rho(S)}\}$. By comonotonic independence, $\geq^*$ is Archimedean, independent, monotonic, and nondegenerate on $C(\rho)$. By applying the argument from the proof of theorem 1.5 to each $C(\rho)$, we can construct a unique collection of measures $\{\mu^\rho \in \Delta(S) : \rho \in \mathcal{P}(S)\}$ such that for all $\rho \in \mathcal{P}(S)$ and all $w, y \in C(\rho)$, $w \geq^* y$ if and only if $\mu^\rho \cdot w \geq \mu^\rho \cdot y$. We now show that for all $w, y \in v(\Delta(X))^S$, $w \geq^* y$ if and only if $\mu^{\hat{\rho}} \cdot w \geq \mu^{\tilde{\rho}} \cdot y$ for any $\hat{\rho}, \tilde{\rho} \in \mathcal{P}(S)$ such that $w \in C(\hat{\rho})$ and $y \in C(\tilde{\rho})$. This follows from the fact that there exist unique $a_w, a_y \in \mathbb{R}$ such that $w =^* \overline{a_w}$ and $y =^* \overline{a_y}$. Therefore, for any $\hat{\rho}, \tilde{\rho} \in \mathcal{P}(S)$ such that $w \in C(\hat{\rho})$ and $y \in C(\tilde{\rho})$, we have $\mu^{\hat{\rho}} \cdot w = \mu^{\hat{\rho}} \cdot \overline{a_w} = a_w$ and $\mu^{\tilde{\rho}} \cdot y = \mu^{\tilde{\rho}} \cdot \overline{a_y} = a_y$. It follows now that $w \geq^* y$ if and only if $a_w \geq a_y$, which is equivalent to $\mu^{\hat{\rho}} \cdot w \geq \mu^{\tilde{\rho}} \cdot y$.

Note that if $E \subset S$ and $1_E \in C(\hat{\rho}) \cap C(\tilde{\rho})$ for some $\hat{\rho}, \tilde{\rho} \in \mathcal{P}(S)$, then $\mu^{\hat{\rho}} \cdot 1_E = \mu^{\tilde{\rho}} \cdot 1_E$ and so $\mu^{\hat{\rho}}(E) = \mu^{\tilde{\rho}}(E)$. Define the mapping $\nu : 2^S \to [0,1]$ by $\nu(E) = \mu^\rho(E)$ for any ρ such that $1_E \in C(\rho)$. Then, ν is well-defined by $\{\mu^\rho : \rho \in \mathcal{P}(S)\}$ and is a capacity since $\nu(\varnothing) = 0$, $\nu(S) = 1$, and if

$E \subset F \subset S$ then $1_E, 1_F$ are comonotonic so that there exists $\hat{\rho} \in \mathcal{P}(S)$ such that $1_E, 1_F \in C(\rho)$ and hence $\nu(E) = \mu^{\hat{\rho}}(E) \leq \mu^{\hat{\rho}}(F) = \nu(F)$.

Let $E \subset S$ and $F = E \cup \{s\}$ for $s \notin E$. For any ρ such that $1_E, 1_F \in C(\rho)$, $\nu(F) = \nu(E) = \mu^\rho(F) - \mu^\rho(E) = \mu^\rho(s)$. Fix $w \in v(\Delta(X))^S$ and ρ such that $w \in C(\rho)$. Then, we have

$$\begin{aligned}\mu^\rho \cdot w = \sum_s \mu^\rho(s) w_s = \sum_s \mu^\rho(\rho(s)) w_{\rho(s)} &= \sum_{s=1}^{S-1} w_{\rho(s)} [\mu^\rho(A_s^\rho) - \mu^\rho(A_{s+1}^\rho)] \\ &= \sum_{s=1}^{S-1} w_{\rho(s)} [\nu(A_s^\rho) - \nu(A_{s+1}^\rho)] = \mathrm{CE}_\nu(w),\end{aligned}$$

proving that for all $w, y \in v(\Delta(X))^S$, $w \geq^* y$ if and only if $\mathrm{CE}_\nu(w) \geq \mathrm{CE}_\nu(y)$.

Finally, if we assume that $\succsim$ is also uncertainty averse, then for all $f, g \in H$ and all $\alpha \in (0,1)$, $f \succsim g$ implies that $\alpha f + (1-\alpha) g \succsim g$. Thus, for all $w, y \in v(\Delta(X))^S$ and all $\alpha \in (0,1)$, $w \geq^* y$ implies that $\alpha w + (1-\alpha) y \geq^* y$. Now, fix $E, F \subset S$. We want to show that ν is a convex capacity, so $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$. Without loss of generality, suppose that $\nu(E) \geq \nu(F)$ and choose $\lambda \geq 1$ such that $\nu(E) = \lambda \nu(F)$. Then, $1_E =^* \lambda 1_F$ and thus $0.5 1_E + 0.5 \lambda 1_F \geq^* \lambda 1_F =^* 1_E$. It follows that

$$\begin{aligned}\nu(E) = \lambda \nu(F) &\leq \mathrm{CE}_\nu(0.5 1_E + 0.5 \lambda 1_F) \\ &= 0.5\, \mathrm{CE}_\nu(1_E + \lambda 1_F) \\ &= 0.5\, \mathrm{CE}_\nu(1_{E \cup F} + (\lambda - 1) 1_F + 1_{E \cap F}).\end{aligned}$$

If we choose $\rho \in \mathcal{P}(S)$ such that $1_{E \cup F}, 1_F, 1_{E \cap F} \in C(\rho)$, then we have

$$\begin{aligned}\nu(E) + \lambda \nu(F) &\leq \mathrm{CE}_\nu(1_{E \cup F} + (\lambda - 1) 1_F + 1_{E \cap F}) \\ &= \mu^\rho(E \cup F) + (\lambda - 1) \mu^\rho(F) + \mu^\rho(E \cap F) \\ &= \nu(E \cup F) + (\lambda - 1) \nu(F) + \nu(E \cap F),\end{aligned}$$

which implies that $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$. Since we have now shown that ν is a convex capacity in the case of uncertainty aversion, a simple application of theorem 3.5 completes the proof.

Lecture 6, February 21: We already have notes that were handed out for lecture 6. It covers the material from Machina and Schmeidler (1995).

4 Preference for Flexibility and Temptation and Self-Control

Lecture 7, February 28: The following discussion is from Kreps (1979). The basic setup consists of a finite set Z of prizes or outcomes and a set of menus M, which is defined as the set of non-empty subsets of Z, $2^Z \backslash \varnothing$. A decision maker faces two stages in this setup: first he chooses a menu $A \in M$ and next he chooses an outcome from this menu $x \in A$. We assume that there is a preference relation $\succsim$ over Z satisfying the standard properties (complete, transitive, etc.), and then note that $\succsim$ can be extended to M by defining $\dot{\succsim}$ on M by

$$A \dot{\succsim} A' \quad \text{if and only if} \quad \forall x' \in A', \exists x \in A \text{ such that } x \succsim x'. \tag{3}$$

It follows that $\dot{\succsim}$ is complete and transitive as well, and that it satisfies

$$A \dot{\succsim} A' \quad \text{if and only if} \quad A \dot{\sim} A \cup A'. \tag{4}$$

Proposition 4.1. *The binary relation $\dot{\succsim}$ on M is complete, transitive, and satisfies* (4) *if and only if there exists a complete and transitive preference relation $\succsim$ on Z such that* (3) *holds.*

The proof of this proposition is left as an exercise. To understand what is meant by preference for flexibility versus temptation and self-control, consider an example. Let $Z = \{s, h, f\}$ so that $M = \{\{s\}, \{h\}, \{f\}, \{s, h\}, \{s, f\}, \{h, f\}, \{s, h, f\}\}$. Suppose that $\{s\} \dot{\succ} \{h\}$. Then, preference for flexibility would imply that $\{s, h\} \dot{\succ} \{s\}, \{h\}$, while temptation and self-control would imply that $\{s\} \dot{\succ} \{s, h\}$. More generally, for any two sets A, A', $A \dot{\succsim} A'$ can be consistent with either $A \cup A' \dot{\succ} A \dot{\succsim} A'$ or $A \dot{\succ} A \cup A'$.

We now develop an axiomatic approach to this issue. For the following axioms, we assume that there is a given preference relation $\dot{\succsim}$ on the set of menus M.

Axiom 4.2. *$\dot{\succsim}$ is a weak order.*

Axiom 4.3. *For any $A, A' \in M$, if $A' \subset A$ then $A \dot{\succsim} A'$.*

Axiom 4.4. *For any $A, A' \in M$, if $A \dot{\sim} A \cup A'$ then for all $A'' \in M$, $A \cup A'' \dot{\sim} A \cup A'' \cup A'$.*

Theorem 4.5. *Let Z be finite. A binary relation $\dot{succsim}$ on $M = 2^Z \backslash \varnothing$ satisfies axioms 4.2-4.4 if and only if there exists a finite set S and a function $U : Z \times S \to \mathbb{R}$ such that*

$$v(A) = \sum_s \left[\max_{x \in A} U(x, s) \right]$$

represents $\dot{\succsim}$ *on* M*.*

It is important to note that the set S and function U representing the preference relation are not necessarily unique. Also, the above expression for $v(A)$ can easily be transformed into an expression of the form $\bar{v}(A) = \sum_s \mu_s[\max_{x\in A} \bar{U}(x,s)]$, where $\mu \in \Delta(S)$.

Lecture 8, March 7: If we let $\dot{\succsim}$ be a weak order on M satisfying axioms 4.3 and 4.4, then we can define the dominance relation $\dot{\geq}$ on M by $A \dot{\geq} A'$ if and only if $A \dot{\succsim} A \cup A'$ for any $A, A' \in M$. With this new definition, we can restate axiom 4.4: if $A \dot{\geq} A'$ and $A \subset A''$, then $A'' \dot{\geq} A'$.

Proposition 4.6. *Let* $\dot{\succsim}$ *be a weak order on* M *satisfying axioms 4.3 and 4.4, and* $\dot{\geq}$ *be the dominance relation* $\dot{\succsim}$ *induces on* M*. Then,*

(a) $\dot{\geq}$ *is reflexive and transitive,*

(b) if $A' \subset A$*, then* $A \dot{\geq} A'$*,*

(c) if $A \dot{\geq} A'$ *and* $A'' \subset A'$*, then* $A \dot{\geq} A''$*,*

(d) if $A_1 \dot{\geq} A_2$ *and* $A_3 \dot{\geq} A_4$*, then* $A_1 \cup A_3 \dot{\geq} A_2 \cup A_4$*,*

(e) for all $A \in M$*, there exists some* $A' \in M$ *such that* $A \subset A'$*,* $A \dot{\geq} A'$*, and* $A \dot{\geq} A''$ *if and only if* $A'' \subset A'$*.*

The proof of this proposition was done for homework. As a consequence of part (e), for each $A \in M$ there exists a maximal set that A dominates, and by part (c) this set is given by

$$f(A) \quad = \bigcup_{\{B \in M : A \,\dot{\geq}\, B\}} B.$$

Also by part (e), it follows that

$$A'' \subset f(A) \quad \text{if and only if} \quad A \dot{\geq} A''. \tag{5}$$

Proposition 4.7. *If* $\dot{\succsim}$ *satisfies axioms 4.2-4.4, then*

(a) for all $A \in M$*,* $f(f(A)) = f(A)$*,*

(b) $A \dot{\geq} A'$ *if and only if* $f(A') \subset f(A)$*.*

For part (a), note that $A \dot{\geq} f(A) \dot{\geq} f(f(A))$ and hence $A \dot{\geq} f(f(A))$. Thus, $f(f(A)) \subset f(A)$ proving that $f(A) = f(f(A))$. For part (b), note first that $A \dot{\geq} A'$ if and only if $A \dot{\geq} f(A')$. This follows because $A \dot{\geq} A' \dot{\geq} f(A')$, and in the other direction because $A \dot{\geq} f(A')$ and $A' \subset f(A')$ together imply that $A \dot{\geq} A'$. But by 5 we have that $A \dot{\geq} f(A')$ if and only if $f(A') \subset f(A)$, completing the proof.

Theorem 4.8. *Let Z be finite. A binary relation $\dot{\succsim}$ on M satisfies axioms 4.2-4.4 if and only if there exists a finite set S, a state-dependent utility $U : Z \times S \to \mathbb{R}$, and a strictly increasing $u : \mathbb{R}^S \to \mathbb{R}$ such that if $w : M \to \mathbb{R}^S$ is defined by*

$$w(A)_s = \max_{x \in A} U(x, s),$$

then $u \circ w$ represents on M.

If $\dot{\succsim}$ satisfies axioms 4.2-4.4, then by theorem 4.5 there exists a function $v : M \to \mathbb{R}$ representing it. Now, let $S = \{A \in M : A = f(A)\}$, and for $(x, s) \in Z \times S$ let

$$U(x, s) = \begin{cases} 1 & \text{if } x \notin s, \\ 0 & \text{if } x \in s. \end{cases}$$

Recall that for all $A \in M$, $f(A) \in S$ by proposition 4.7. With the above definitions, it follows that

$$w(A)_s = \begin{cases} 1 & \text{if } A \not\subset s, \\ 0 & \text{if } A \subset s. \end{cases}$$

We claim that for any $A \in M$, $w(A) \geq w(A')$ if and only if $A \dot{\geq} A'$. First, if $A \dot{\ngeq} A'$, then $A' \notin f(A)$ so that $w(A')_{f(A)} = 1 > 0 = w(A)_{f(A)}$ and hence $w(A) \ngeq w(A')$. Conversely, if $A \dot{\geq} A'$, then any $s \in S$ such that $w(A)_s = 0$ has $A \subset s$. But this implies that $s \dot{\geq} A$ so that $f(A) \subset f(s) = s$ and also $f(A') \subset f(A) \subset s$. Of course, $A' \subset f(A')$ and hence we have that $w(A')_s = 0$, proving that $w(A) \geq w(A')$.

Now, we can define $u : \mathbb{R}^S \to \mathbb{R}$ by setting $u(w(A)) = v(A)$ where $A \in M$. To see that this is well-defined note that if $w(A) = w(A')$ then $A \dot{\geq} A' \dot{\geq} A$, so that $A \dot{\sim} A \cup A' \dot{\sim} A'$ and hence $v(A) = v(A \cup A') = v(A')$. Also, u is strictly increasing. If $w(A) > w(A')$, then $A \dot{\geq} A'$ and $A' \dot{\ngeq} A$, so that $A \dot{\sim} A \cup A'$ and $A' \dot{\nsim} A \cup A'$. But then we have $A \cup A' \dot{\succ} A'$, which implies that $v(A) = v(A \cup A') > v(A')$ and thus $u(w(A)) > u(w(A'))$.

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